CRITICAL POINTS AND SUPERSYMMETRIC VACUA

MICHAEL R. DOUGLAS, BERNARD SHIFFMAN, AND STEVE ZELDITCH

Abstract. Supersymmetric vacua ('universes') of string/M theory may be identified with certain critical points of a holomorphic section (the 'superpotential') of a Hermitian holomorphic line bundle over a complex manifold. An important physical problem is to determine how many vacua there are and how they are distributed. The present paper initiates the study of the statistics of critical points $\nabla s = 0$ of Gaussian random holomorphic sections with respect to a connection ∇. Even the expected number of critical points depends on the curvature of ∇. The principal results give formulas for the expected density and number of critical points of Gaussian random sections relative to ∇ in a variety of settings. The results are particularly concrete for Riemann surfaces. Analogous results on the density of critical points of fixed Morse index are given.

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1. Introduction

In a recent series of articles, one of the authors (M. R. Douglas) has initiated a program to study the vacuum selection problem of string/M theory from a statistical point of view [Do, AD]. This is the problem of finding which (if any) solutions of the theory describe our universe. At present, we know very little about the nature of this problem: how many solutions there are, how many solutions we should expect to fit present-day observations, the range of further predictions these solutions make, and so forth. But, we are reaching the point where such questions can be studied systematically.

We briefly summarize the standard paradigm, called compactification. String theory is formulated in ten space-time dimensions. To describe our universe, one considers a solution which is a direct product of four dimensional Minkowski space-time, with a compact six dimensional manifold (say a Calabi-Yau manifold with metric) carrying additional structures (a vector bundle and other discrete choices). Given such a solution, one can derive an "effective field theory," in which the continuous parameters of the solution (such as the complex structure moduli of the Calabi-Yau) become fields in four dimensional space-time. One then looks for "vacuum configurations" (or simply "vacua") of this field theory, meaning stable time-independent classical solutions; each is a possible candidate to describe our universe. The physical properties of a vacuum (such as masses of particles) are then obtained by the analysis of small fluctuations around the solution.

In the best studied (and perhaps most plausible) examples, the effective field theory is an "$N = 1$ supergravity theory" [WB]. A primary datum in this theory is a *superpotential* W, a holomorphic section of a line bundle $\mathcal{L} \to \mathcal{M}_{CY}$ over the moduli space of Calabi-Yau manifolds. A large class of vacuum configurations, the supersymmetric vacua, are the critical points of this superpotential. Thus, part of the problem of counting vacua is to count critical points of a given holomorphic section.

Now at present there is no computation of an exact superpotential for any string compactification. Thus, to get insight into general features of this problem, one might model the superpotential as a random holomorphic section of $\mathcal{L}$, much as the potential in a disordered system is regarded as random, and study the statistics of vacua (critical points) of a random superpotential.

In setting up a statistical model of 'random superpotentials', one must decide which probability measure to put on the space of candidates. From many points of view, the simplest candidate is to take Gaussian random holomorphic sections in $H^0(\mathcal{M}_{CY}, \mathcal{L})$ with respect to a certain covariance kernel. This connects the string/M problem with the statistical theory of holomorphic sections developed by the other two authors in collaboration with P. Bleher in [SZ, BSZ1, BSZ2].

A further point in favor of this probability measure, is that it can be used to get physical results. Now there are known superpotentials which become exact upon taking partial limits in moduli space; for example taking the volume of the Calabi-Yau metric large, at arbitrary complex structure. A particularly interesting class of these are the "flux superpotentials" which are linear combinations from a computable basis of sections. As argued in [AD], an asymptotic estimate for the number of flux vacua in a family of compactifications constructed by Giddings, Kachru and Polchinski [GKP] can be obtained from the expected number of critical points in a Gaussian ensemble of superpotentials.

We discuss the physical background of our problem in more depth in section 2, and for the remainder of the introduction, concentrate on the mathematical results.

From the mathematical viewpoint, the statistical theory of critical points of Gaussian random holomorphic sections depends on the following objects:

- A choice of subspace $\mathcal{S} \subset H^0(M, L)$ of holomorphic sections of a holomorphic line bundle $L \to M$. We assume $\mathcal{S}$ to be finite dimensional, but our methods and results extend easily to infinite dimensional spaces of sections.
- A choice of Gaussian measure γ on $\mathcal{S}$, or equivalently an inner product $\langle , \rangle$ on $\mathcal{S}$.
- A choice of Hermitian metric h on L. This gives rise to the Chern connection ∇ on L, which is of type $(1, 0)$ with curvature of type $(1, 1)$.

The physical application requires a rather general framework of $(\mathcal{S}, \gamma, \nabla)$. Our purpose is to study the distribution and number of critical points

$$\nabla s(z) = 0, \quad s \in \mathcal{S} \tag{1}$$

of a random section $s \in \mathcal{S}$ with respect to γ. Since critical points are zeros of random sections $\nabla s \in \nabla\mathcal{S} \subset \mathcal{C}^\infty(T^*M \otimes L)$, we are able to use the previous work [SZ, BSZ1, BSZ2] on zeros of random $\mathcal{C}^\infty$ sections of complex vector bundles to set up the statistical theory. It is important to observe that the critical point equation (1) is not holomorphic, and therefore the much simpler statistical theory of zeros of holomorphic sections in [SZ, BSZ1, BSZ2] does not apply. On the other hand, the holomorphicity of the original sections permeates the calculations, and gives the statistics of their critical points some special features which do not hold in the general $\mathcal{C}^\infty$ case. The purpose of this article is to develop a self-contained theory of critical points of Gaussian random holomorphic sections which makes use of these special features. It is not necessary to know the results of [BSZ1, BSZ2] to read this paper. At the risk of being repetitious, we have tried to make this article more accessible to physicists as well as mathematicians by sometimes giving two proofs of the same assertion or two explanations of a key idea in both mathematical and physical language.

The key object of interest in this article is the *expected distribution of critical points* of a Gaussian random holomorphic section $s \in \mathcal{S} \subset H^0(M, L)$. The inner product on $\mathcal{S}$ determines a Gaussian probability measure γ (see Definition 4.1). As discussed in §3, the definition $\nabla s = 0$ of critical point depends on a choice of connection, which we always choose to be the Chern (Hermitian) connection associated to a Hermitian metric h on L. For almost any section $s \in H^0(M, L)$, the set of its critical points $Crit^\nabla(s)$ is discrete and we define the distribution of critical points of s to be the (un-normalized) measure

$$C^\nabla_s := \sum_{z \in Crit^\nabla(s)} \delta_z, \tag{2}$$

where δ_z is the Dirac point mass at z. We let

$$\mathbf{K}^{\mathrm{crit}}_{\mathcal{S},\gamma,\nabla} = \mathbf{E}\,_\gamma C^\nabla_s \tag{3}$$

denote the expected distribution of critical points, i.e. the average of the measures C^∇_s with respect to $\gamma_{\mathcal{S}}$. If M is compact, the expected total number of critical points is then given by

$$\mathcal{N}^{\mathrm{crit}}(\mathcal{S}, \gamma, \nabla) = \mathbf{K}^{\mathrm{crit}}_{\mathcal{S},\gamma,\nabla}(M)\ . \tag{4}$$

Our first result is a formula for the expected critical point distribution $\mathbf{K}^{\mathrm{crit}}_{\mathcal{S},\gamma,\nabla}$, which is valid for any subspace $\mathcal{S} \subset H^0(M,L)$ of holomorphic sections of any holomorphic line bundle over any complex manifold (possibly non-compact and/or incomplete). We shall assume that the space $\mathcal{S}$ of sections satisfies a technical condition, the *2-jet spanning property*, which says that all possible values and derivatives of order ≤ 2 are attained by the global sections $s \in \mathcal{S}$ at every point of M (see Definition 5.1). In particular, the formula for $\mathbf{K}^{\mathrm{crit}}_{\mathcal{S},\gamma,\nabla}$ applies to the physically relevant case in string/M theory where $L \to M$ is a negative line bundle over the moduli space of Calabi-Yau manifolds (an incomplete, non-compact Kähler manifold), where $\mathcal{S}$ is a special subspace of sections given by periods of the Calabi-Yau form, and where γ is induced by a rather subtle inner product coming from the intersection form on cycles.

To state the result, it is most convenient to introduce a local frame (non-vanishing holomorphic section) e_L for L and local coordinates $(z_1,\ldots,z_m)$ on an open set $U \subset M$, and to write

$$\mathbf{K}^{\mathrm{crit}}_{\mathcal{S},\gamma,\nabla} = \mathbf{k}^{\mathrm{crit}}_{\mathcal{S},\gamma,\nabla}\, dz\ , \tag{5}$$

where $dz = \prod_{j=1}^m \left(\frac{i}{2} dz_j \wedge d\bar{z}_j\right)^m$ is Lebesgue measure with respect to these coordinates. (Note that $\mathbf{k}^{\mathrm{crit}}_{\mathcal{S},\gamma,\nabla}$ depends on the coordinates.) We denote the curvature $(1,1)$-form of ∇ in these coordinates by $\Theta = \sum_{j,k=1}^m \Theta_{jk} dz^j \wedge d\bar{z}^k$ and refer to the $m \times m$ matrix $\left(\Theta_{jk}\right)$ as the curvature matrix of ∇. We also denote by $\mathrm{Sym}(m,\mathbb{C})$ the space of complex $m \times m$ symmetric matrices. It is a Hermitian vector space with inner product $\langle A, B\rangle = Tr AB^*$. We also consider the Hermitian orthogonal sum $\mathrm{Sym}(m,\mathbb{C}) \oplus \mathbb{C}$ with the standard Hermitian inner product on $\mathbb{C}$. These induce a natural volume form on $\mathrm{Sym}(m,\mathbb{C}) \oplus \mathbb{C}$. Finally, the *covariance kernel* or two-point function $\Pi_{\mathcal{S}}(z,w)$ of the Gaussian measure γ is defined in Definition 4.2.

Theorem 1. *Let $(\mathcal{S},\gamma,\nabla)$ denote a finite-dimensional subspace $\mathcal{S} \subset H^0(M,L)$ of holomorphic sections of a holomorphic line bundle $L \to M$ with a Chern connection ∇ on an m-dimensional complex manifold, together with a Gaussian measure γ on $\mathcal{S}$. Assume that $\mathcal{S}$ satisfies the 2-jet spanning property. Given local coordinates $z = (z_1,\ldots,z_m)$ and a local frame e_L for L, there exist positive-definite Hermitian matrices*

$$A(z) : \mathbb{C}^m \to \mathbb{C}^m\ , \quad \Lambda(z) : \mathrm{Sym}(m,\mathbb{C}) \oplus \mathbb{C} \to \mathrm{Sym}(m,\mathbb{C}) \oplus \mathbb{C}\ ,$$

depending only on z, ∇ and $\Pi_{\mathcal{S}}$ (cf. (6)–(9)) such that the expected density of critical points with respect to Lebesgue measure dz is given by

$$\begin{aligned}\mathbf{k}^{\mathrm{crit}}_{\mathcal{S},\gamma,\nabla}(z) \quad &= \quad \frac{1}{\pi^{\binom{m+2}{2}} \det A(z) \det \Lambda(z)} \\ &\qquad \times \int_{\mathbb{C}} \int_{\mathrm{Sym}(m,\mathbb{C})} \left| \det \begin{pmatrix} H' & x\,\Theta(z) \\ \bar{x}\,\bar{\Theta}(z) & \bar{H}' \end{pmatrix} \right| e^{-\left\langle \Lambda(z)^{-1}(H' \oplus x),\, H' \oplus x \right\rangle} dH'\, dx\ ,\end{aligned}$$

where $\Theta(z)$ is the curvature matrix of ∇ in the coordinates $(z_1,\ldots,z_m)$.

The matrix in the formula is the complex Hessian of s discussed in §3.1.

In order to give the simplest expressions for the matrices $A(z)$ and $\Lambda(z)$, we let e_L be an *adapted* local frame at a point $z_0 \in M$; i.e., e_L has the property that the pure holomorphic derivatives (of order ≤ 2) of the local connection form for ∇ vanish at z_0 (see Definition 3.2). We then let $F_{\mathcal{S}}(z,w)$ be the local expression for $\Pi_{\mathcal{S}}(z,w)$ in the frame e_L (see Definition

5.3). Then

$$A(z_0) = \left(\frac{\partial^2}{\partial z_j \partial \bar{w}_{j'}} F_{\mathcal{S}}(z,w)\Big|_{(z,w)=(z_0,z_0)} \right) \tag{6}$$

and

$$\Lambda(z_0) = C(z_0) - B(z_0)^* A(z_0)^{-1} B(z_0)\ , \tag{7}$$

where

$$B(z_0) \;=\; \left[\left(\frac{\partial^3}{\partial z_j \partial \bar{w}_{q'} \partial \bar{w}_{j'}} F_{\mathcal{S}}(z,w) \right) \quad \left(\frac{\partial}{\partial z_j} F_{\mathcal{S}}(z,w) \right) \right]\Big|_{(z,w)=(z_0,z_0)}\ , \tag{8}$$

$$C(z_0) \;=\; \left[\begin{array}{cc} \left(\frac{\partial^4}{\partial z_q \partial z_j \partial \bar{w}_{q'} \partial \bar{w}_{j'}} F_{\mathcal{S}}(z,w) \right) & \left(\frac{\partial^2}{\partial z_j \partial z_q} F_{\mathcal{S}}(z,w) \right) \\ \left(\frac{\partial^2}{\partial \bar{w}_{q'} \partial \bar{w}_{j'}} F_{\mathcal{S}}(z,w) \right) & F_{\mathcal{S}}(z,z) \end{array} \right]\Bigg|_{(z,w)=(z_0,z_0)}\ , \tag{9}$$

$$1 \le j \le m\,, 1 \le j \le q \le m\,, 1 \le j' \le q' \le m\,.$$

In the above, A, B, C are $m\times m$, $m\times n$, $n\times n$ matrices, respectively, where $n = \frac{1}{2}(m^2+m+2)$. We also provide formulas for these matrices in a non-adapted local frame (cf. (74)–(76)), which are useful when one studies variations of the critical point distribution with respect to ∇.

Of course, the expected distribution of zeros $\mathcal{K}^{\mathrm{crit}}_{\mathcal{S},\gamma,\nabla}(z)dz$ is independent of the choice of frame and coordinates. One can write the formula in an invariant form by interpreting

$$A(z) : T^{*1,0}_{M,z} \otimes L_z \to T^{*1,0}_{M,z} \otimes L_z\ , \quad \Lambda(z) : (S^2 T^{*1,0}_{M,z} \oplus \mathbb{C}) \otimes L_z \to (S^2 T^{*1,0}_{M,z} \oplus \mathbb{C}) \otimes L_z$$

as positive-definite Hermitian operators, where $S^2 T^{*1,0}_{M,z} \subset T^{*1,0}_{M,z} \otimes T^{*1,0}_{M,z}$ denotes the symmetric product, $\Theta(z) \in T^{*1,0}_{M,z} \otimes T^{*0,1}_{M,z}$ is the curvature operator of ∇, and the determinant in the integral is an element of $(\det T^*_{M,z})^2 \otimes L^m_z$. However, we find the local expressions to be more useful.

Since the integral contains an absolute value, it is difficult to evaluate the density explicitly when the dimension is greater than 1, or even to analyze its dependence on ∇, γ. In particular, one cannot simplify it with Wick's formula.

A special case of geometric interest is where the inner product $\langle , \rangle$ and Gaussian measure on $\mathcal{S}$ are induced by a volume form dV on M and the same Hermitian metric h which determines ∇, namely

$$\langle s_1, s_2 \rangle = \int_M h_z\big(s_1(z), s_2(z)\big)\, dV(z)\,. \tag{10}$$

The covariance kernel is then the Szegö kernel of $\mathcal{S}$, i.e. the orthogonal projection

$$(\Pi_{\mathcal{S},h,V}\, s)\,(z) = \int_M h_w\big(s(w), \Pi_{\mathcal{S},h,V}(z,w)\big) dV(w)\,, \qquad \Pi_{\mathcal{S},h,V}(z,w) \in L_z \otimes \bar{L}_w\ . \tag{11}$$

We refer to this Gaussian measure as the *Hermitian Gaussian measure* on $\mathcal{S}$. In this case, every object in the density of critical points is determined by the metric and volume form, and we have a direct relation between the expected number and distribution of critical points and the metric.

The simplest metric situation is that of positive line bundles. In this case, we assume that $\omega = \frac{i}{2}\Theta_h$ so that $c_1(L) = [\frac{1}{\pi}\omega]$, where the brackets denote the cohomology class. More precisely, the Kähler form is given by

$$\omega = \frac{i}{2}\Theta_h = \frac{i}{2}\partial\bar\partial K, \qquad K = -\log |e_L|_h^2.$$

The volume form is then assumed to be

$$dV = \frac{\omega^m}{m!}$$

(and thus the total volume of M is $\frac{\pi^m}{m!}c_1(L)^m$). If L is a negative line bundle (on a noncompact manifold M), we instead choose the Kähler form $\omega = -\frac{i}{2}\Theta_h$. We shall write

$$\mathbf{K}^{\rm crit}_{\mathcal{S},\gamma,\nabla} = \mathcal{K}^{\rm crit}_{\mathcal{S},h}\, dV \ , \quad \mathcal{N}^{\rm crit}(\mathcal{S},\gamma,\nabla) = \mathcal{N}^{\rm crit}(\mathcal{S},h), \tag{12}$$

where γ is the Hermitian Gaussian measure described above. (Note that $\mathcal{K}^{\rm crit}_{\mathcal{S},h}$ denotes the density with respect to the volume dV, while $\mathbf{k}^{\rm crit}_{\mathcal{S},\gamma,\nabla}$ is the density with respect to Lebesgue measure in local coordinates.) As a consequence of Theorem 1, we obtain the following integral formula for the critical point density $\mathcal{K}^{\rm crit}_{\mathcal{S},h}$ in these cases:

Corollary **2.** *Let $(L,h) \to M$, $(\mathcal{S},\gamma,\nabla)$ be as in Theorem 1. Further assume that the curvature form of Θ_h is either positive or negative and that γ is the Hermitian Gaussian measure* (10) *induced by h and by the volume form $dV = \frac{1}{m!}\left(\pm\frac{i}{2}\Theta_h\right)^m$. Then the expected density of critical points relative to dV is given by*

$$\mathcal{K}^{\rm crit}_{\mathcal{S},h}(z) = \frac{\pi^{-\binom{m+2}{2}}}{\det A(z)\det\Lambda(z)}\int_{{\rm Sym}(m,\mathbb{C})\times\mathbb{C}} \left|\det(H'H'^* - |x|^2 I)\right| e^{-\langle \Lambda(z)^{-1}(H',x),(H',x)\rangle}\, dH'\, dx\,.$$

where $A(z),\Lambda(z)$ are positive Hermitian matrices (depending on h and z) given by (6)*–*(9)*.*

As above, $H' \in {\rm Sym}(m,\mathbb{C})$ is a complex symmetric matrix, and the matrix Λ is a Hermitian operator on the complex vector space ${\rm Sym}(m,\mathbb{C})\times\mathbb{C}$. The only point of the corollary is that we can identify $\Theta = I$ in normal coordinates and simplify the determinant.

As mentioned above, the physically relevant case is that of a negative line bundle over an incomplete Kähler manifold (the moduli space of Calabi-Yau metrics on a 3-fold M). As mentioned above, the relevant Gaussian measure there is not the Hermitian one. However, the same formula holds in that case since the curvature form equals $-I$ in local coordinates adapted to the Weil-Petersson volume form.

In dimension one, we obtain the following explicit formula for the expected density of critical points in terms of the eigenvalues of ΛQ_r, where

$$Q_r = \begin{pmatrix} 1 & 0 \\ 0 & -r^2 \end{pmatrix}\ ,$$

and $r = \frac{i}{2}\Theta_h/dV$:

Theorem **3.** *Let $(L,h) \to M$ be a Hermitian line bundle on a (possibly non-compact) Riemann surface M with area form dV. Let $\mathcal{S}$ be a finite-dimensional subspace of $H^0(M,L)$ with the 2-jet spanning property, and let γ be the induced Hermitian Gaussian measure. Let*

$\mu_1 = \mu_1(z)$, $\mu_2 = \mu_2(z)$ denote the eigenvalues of $\Lambda(z)Q_r$, where $r = r(z) = \frac{i}{2}\Theta_h/dV$. Then $\mathbf{K}^{\mathrm{crit}}_{\mathcal{S},\gamma,\nabla} = \mathcal{K}_{\mathcal{S},h,V}\,dV$, where

$$\mathcal{K}^{\mathrm{crit}}_{\mathcal{S},h,V} = \frac{1}{\pi A}\,\frac{\mu_1^2+\mu_2^2}{|\mu_1|+|\mu_2|} = \frac{1}{\pi A}\,\frac{Tr\,\Lambda^2}{Tr|\Lambda^{\frac12}Q_r\Lambda^{\frac12}|}\ ,$$

where A, Λ are given by (6)–(9).

We define the *topological index* of a section s at a critical point z_0 to be the index of the vector field ∇s at z_0 (where ∇s vanishes). Critical points of a section s in dimension one are (almost surely) of topological index ± 1. (If the connection were flat, then ∇s would be holomorphic and the topological indices would all be positive.) The critical points of s of index 1 are the saddle points of $\log|s|_h$ (or equivalently, of $|s|_h^2$), while those of topological index -1 are local maxima of $\log|s|_h$ in the case where L is positive, and are local minima of $|s|_h^2$ if L is negative. (If L is negative, the length $|s|_h$ cannot have local maxima; if L is positive, the only local minima of $|s|_h$ are where s vanishes.) Thus, in dimension 1, topological index 1 corresponds to $\log|s|_h$ having Morse index 1, while topological index -1 corresponds to Morse index 2 if L is positive. In fact, in all dimensions, the critical points of a section s are the critical points of $\log|s|_h$, and for positive line bundles L, we have

$$\mathrm{index}_{z_0}(\nabla s) = (-1)^{m+\text{Morse index}_{z_0}(\log|s|)}\ , \tag{13}$$

at (nondegenerate) critical points z_0 (see Lemma 7.1).

From the proof of Theorem 3 we obtain:

Corollary **4.** *Let $(L,h)\to(M,dV)$, μ_1,μ_2 be as in Theorem 3. Then:*

- *The expected density of critical points of topological index 1 (where $|s|_h^2$ has a saddle point) is given by*

$$\mathcal{K}^{\mathrm{crit}}_{+}(z) = \frac{1}{\pi A(z)}\,\frac{\mu_1^2}{|\mu_1|+|\mu_2|}\ ,$$

- *The expected density of critical points of topological index -1 (where $|s|_h^2$ has a local maximum) is*

$$\mathcal{K}^{\mathrm{crit}}_{-}(z) = \frac{1}{\pi A(z)}\,\frac{\mu_2^2}{|\mu_1|+|\mu_2|}\ .$$

- *Hence, the index density is given by*

$$\mathcal{K}^{\mathrm{crit}}_{\mathrm{index}} := \mathcal{K}^{\mathrm{crit}}_{+}(z) - \mathcal{K}^{\mathrm{crit}}_{-}(z) = \frac{1}{\pi A(z)}\,(\mu_1+\mu_2) = \frac{1}{\pi A(z)}\,Tr[\Lambda(z)Q]\ .$$

The simplest case of Theorem 3 and Corollary 4 is that of sections of powers $\mathcal{O}(N)$ of the hyperplane line bundle $\mathcal{O}(1)\to\mathbb{CP}^1$, i.e. of homogeneous polynomials of degree N in the $SU(2)$ ensemble, where we can give exact formulas:

Corollary **5.** *The expected numbers $\mathcal{N}^{\mathrm{crit}}_{N,+}$ and $\mathcal{N}^{\mathrm{crit}}_{N,-}$ of critical points of topological index 1 and -1, respectively, of a random section $s_N\in H^0(\mathbb{CP}^1,\mathcal{O}(N))$ (endowed with the Hermitian Gaussian measure induced from the Fubini-Study metrics h^N on $\mathcal{O}(N)$ and ω_{FS} on $\mathbb{CP}^1$) are*

given by

$$\begin{aligned}\mathcal{N}^{\rm crit}_{N,+} &= \frac{4(N-1)^2}{3N-2} = \frac{4}{3}\,N - \frac{16}{9} + \frac{4}{27}\,N^{-1}\cdots \quad (\textit{number of saddle points of } |s|^2_h)\ ,\\ \mathcal{N}^{\rm crit}_{N,-} &= \frac{N^2}{3N-2} = \frac{1}{3}\,N + \frac{2}{9} + \frac{4}{27}\,N^{-1}\cdots \quad (\textit{number of local maxima of } |s|^2_h)\ ,\end{aligned}$$

and thus the expected total number of critical points is given by

$$\mathcal{N}^{\rm crit}_N(\mathbb{CP}^1) = \frac{5N^2-8N+4}{3N-2} = \frac{5}{3}\,N - \frac{14}{9} + \frac{8}{27}\,N^{-1}\cdots\ .$$

It follows that the average number of critical points of a polynomial $p(z)$ of degree $N>1$ in the $SU(2)$ ensemble is greater for all N than the almost sure number of critical points ($=N-1$) in the classical sense of $p'(z)=0$. This is not surprising, since in the former case, sections may have critical points of index -1 and in the latter case there are no critical points of index -1, while $\mathcal{N}^{\rm crit}_{N,+} - \mathcal{N}^{\rm crit}_{N,-} = c_1\left(\mathcal{O}(N)\otimes K_{\mathbb{CP}^1}\right) = N-2$. (The number of critical points in the latter case is $N-1$ instead of $N-2$, since $p'(z)\,dz$ almost surely has a pole of order 1 at ∞.) In an asymptotic sense, there are $\frac{5}{3}$ as many critical points in the metric sense.

In higher dimensions, the integral in Corollary 2 is more complicated to evaluate, and the density does not have a simple formulation in terms of eigenvalues as in Theorem 3. In our subsequent paper [DSZ], we derive the following alternate formula for the expected density of critical points:

$$\mathcal{K}^{\rm crit}_{\mathcal{S},h}(z) = \frac{c_m}{\det A(z)} \lim_{\varepsilon,\varepsilon'\to 0^+} \int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\int_{\mathrm{U}(m)} \frac{\Delta(\xi)\,\Delta(\lambda)\,\left|\prod_j \lambda_j\right| e^{i\langle\xi,\lambda\rangle} e^{-\epsilon|\xi|^2-\epsilon'|\lambda|^2}}{\det\left[i\widehat{D}(\xi)\rho(g)\Lambda(z)\rho(g)^*+I\right]}\,dg\,d\xi\,d\lambda,$$

where

- $c_m = \dfrac{(-i)^{m(m-1)/2}}{2^m\,\pi^{2m}\,\prod_{j=1}^m j!}$,
- $\widehat{D}(\xi)$ is the Hermitian operator on $\mathrm{Sym}(m,\mathbb{C})\oplus\mathbb{C}$ given by
$$\widehat{D}(\xi)\big((H_{jk}),x\big) = \left(\left(\frac{\xi_j+\xi_k}{2}\,H_{jk}\right),\ -\left(\textstyle\sum_{q=1}^m \xi_q\right)x\right)\ ,$$
- ρ is the representation of $\mathrm{U}(m)$ on $\mathrm{Sym}(m,\mathbb{C})\oplus\mathbb{C}$ given by
$$\rho(g)(H,x) = (gHg^t, x)\ .$$

We use this formula in [DSZ] to compute (with the assistance of Maple) the expected numbers $\mathcal{N}^{\rm crit}_N(\mathbb{CP}^m)$ of critical points of sections of the N-th power $\mathcal{O}(N)$ of the hyperplane section bundle on $\mathbb{CP}^m$, for $m\le 4$. For example, for the projective plane, we have

$$\mathcal{N}^{\rm crit}_N(\mathbb{CP}^2) = \frac{59\,N^5 - 231\,N^4 + 375\,N^3 - 310\,N^2 + 132\,N - 24}{(3\,N-2)^3} \sim \frac{59}{27}\,N^2\ . \tag{14}$$

This expected number is with respect to the $SU(3)$-invariant geometry on $\mathcal{O}(N)\to\mathbb{CP}^2$. We conjecture that (14) is the minimum expected number of critical points over all connections on $\mathcal{O}(N)\to\mathbb{CP}^2$ with positive curvature, and we show in [DSZ] that this is the case in an asymptotic sense.

In our sequel [DSZ], we also obtain asymptotic results on the expected numbers of critical points of $f = |s|_h^2$ of each possible Morse index for powers L^N of a positive line bundle (L,h) over any compact complex manifold. In particular, for a positive line bundle (L,h) over a compact Riemann surface C endowed with the Kähler form $\omega_h = \frac{i}{2}\Theta_h$, we prove that

$$\mathcal{N}^{\rm crit}_{N,+} = \frac{4}{3}\,c_1(L)\,N + \frac{8}{9}\,(2g-2) + \left(\frac{1}{27\pi}\int_C \rho^2\omega_h\right) N^{-1} + O(N^{-2})\,, \tag{15}$$

$$\mathcal{N}^{\rm crit}_{N,-} = \frac{1}{3}\,c_1(L)\,N - \frac{1}{9}\,(2g-2) + \left(\frac{1}{27\pi}\int_C \rho^2\omega_h\right) N^{-1} + O(N^{-2})\,, \tag{16}$$

where g denotes the genus of C and ρ is the Gaussian curvature of (C,ω_h). Thus, the expected number of local maxima of $|s|_h$ (on Riemann surfaces of any genus) is $\sim \frac{1}{3}c_1(L)N \sim \frac{1}{3}\dim H^0(M,L^N)$. It would be interesting to find a heuristic reason for the factor $\frac{1}{3}$.

It is well known that on complex manifolds M of dimension m, critical points of a section $s \in H^0(M,L)$ are critical points of the function $\log|s|_h$ (and conversely) and these have Morse index $\geq m$ when L has positive curvature (see [Bo]). (Recall that the Morse index of a nondegenerate critical point of a real-valued function is the number of negative eigenvalues of its Hessian matrix.) In this case, we have a density formula for critical points of any Morse index:

THEOREM **6.** *Let $(L,h) \to M$ be a positive holomorphic line bundle over a complex manifold M with volume form $dV = \frac{1}{m!}(\frac{i}{2}\Theta_h)^m$. Suppose that $H^0(M,L)$ contains a finite-dimensional subspace $\mathcal{S}$ with the 2-jet spanning property, and let γ be the Hermitian Gaussian measure on $\mathcal{S}$. Then the expected density with respect to dV of critical points of* $\log|s|_h$ *of Morse index q is given by*

$$\mathcal{K}^{\rm crit}_{\mathcal{S},h,q}(z) = \frac{\pi^{-\binom{m+2}{2}}}{\det A(z)\det\Lambda(z)} \int_{\mathbf{S}_{m,q-m}} \left|\det(SS^* - |x|^2 I)\right| e^{-\langle \Lambda(z)^{-1}(S,x),(S,x)\rangle}\, dS\, dx\,.$$

where

$$\mathbf{S}_{m,k} = \{S \in {\rm Sym}(m,\mathbb{C}) \times \mathbb{C} : {\rm index}(SS^* - |x|^2 I) = k\}\,.$$

This article is just the first in a series and leaves many issues unexplored. For simplicity let us assume that $\mathcal{S} = H^0(M,L)$ and drop it from the notation. First, it would be interesting to explore the dependence of the density and expected number $\mathcal{N}^{\rm crit}(\nabla,\gamma)$ of critical points on the connection ∇ and the Gaussian measure γ. In §6.4.1 we give a simple proof that $\mathcal{N}^{crit}(\nabla,\gamma)$ is non-constant in (∇,γ). The number $\mathcal{N}^{\rm crit}(\nabla,\gamma)$ might be viewed as defining a configurational entropy for statistics of vacua. It is bounded below by the Euler characteristic $c_m(L\otimes K_M)$, but (as the argument in §6.4.1 suggests) is probably not bounded above. It would be interesting to prove this, and to analyze how $\mathcal{N}^{\rm crit}(\nabla,\gamma)$ depends on the choice of ∇,γ? Does there exist a smooth ∇ with a minimal average number of critical points? (Clearly, the meromorphic ∇ gives the minimal number for $\mathcal{O}(N) \to \mathbb{CP}^m$, but the number jumps as one moves from a smooth to a meromorphic connection.)

These problems become purely geometrical when the Gaussian measure is Hermitian, i.e. $\gamma = \gamma_{\mathcal{S},h,V}$. In this case, does there exist a metric for which $\mathcal{N}^{\rm crit}(h)$ is minimal? Is it unique? The formulas above express $\mathcal{N}^{\rm crit}(h)$ in terms of the metric Szegö kernel and are not explicit in terms of the geometry of (L,h). How does the curvature of h influence $\mathcal{N}^{\rm crit}(h)$. Is it bounded in a set of metrics with curvature bounds? Do positively curved metrics on L have

larger $\mathcal{N}^{\mathrm{crit}}(h)$ than signed curvature ones, as suggested by the uniform distribution result? Can one link the expected number of critical points for meromorphic connections with that for smooth connections?

In subsequent work [DSZ], we will analyze the asymptotics of the density and number $\mathcal{N}^{\mathrm{crit}}(h^N)$ of critical points for powers $(L^N, h^N) \to (M, \omega)$ of a positive Hermitian line bundle $(L, h) \to (M, \omega)$. We will show that, as in Theorem 5, the density $\mathcal{K}^{\mathrm{crit}}_{N,h}$ of critical points has a complete asymptotic expansion in N, whose leading coefficient is a universal constant with respect to the curvature volume form $\frac{\omega^m}{m!}$. Thus, critical points become uniformly distributed with respect to the curvature volume form. Furthermore, we will analyze the asymptotic dependence of the expected number of critical points $\mathcal{N}(h^N)$ on the metric. But this asymptotic study does not seem to answer the above questions on a fixed positive line bundle.

Finally, the motivating problem is that of statistics of vacua in string/M theory. In this case, the number $\mathcal{N}^{\mathrm{crit}}(\mathcal{S}, \nabla, \gamma)$ depends on the particular choice of the ample subspace $\mathcal{S}$ of periods and on the special choice of γ coming from the intersection form. Moreover, the Gaussian measure is only an approximation to the discrete probability space of periods. Ultimately, we would like to understand the above questions in this setting.

2. Physical Background

In this section, we give precise definitions for the physical theories we study. As stated in the introduction, we do not discuss string or M theory directly, but rather assume that a given string or M theory compactification corresponds to an "effective $\mathcal{N} = 1$ supergravity theory," in a way we sketch in an example below.

The standard references for supergravity and other field theories with "$\mathcal{N} = 1$ supersymmetry" are [WB, We], and nice treatments of supersymmetry for mathematicians are [Fr1, IAS]. Field theories are usually defined by specifying an action functional, which is written in terms of fields which are sections of various spinor and tensor bundles over $\mathbb{R}^{D,1}$, taking values in a configuration space M and its associated bundles.

For present purposes, the basic data specifying a supergravity theory T is a triple (M, K, W), where

- M is the "configuration space," a complex Kähler manifold. We will typically denote its dimension as d, and local complex coordinates as z^i. We will also refer to these coordinates as "fields."
- K is the Kähler potential, determining the metric on M.
- W is the superpotential, a holomorphic section (possibly with singularities) of the associated line bundle $\mathcal{L}$ with $c_1(\mathcal{L}) = -\frac{1}{\pi}\omega$, where the Kähler form $\omega = \frac{i}{2}\partial\bar{\partial}K$.

Such an associated line bundle carries a natural holomorphic connection whose curvature is the Kähler form. It is the connection which preserves the Hermitian metric on the fibers

$$||W||^2 = e^K |W|^2;$$

explicitly, the covariant derivative of a section W is

$$D_i W \equiv \partial_i W + (\partial_i K) W, \tag{17}$$

while the curvature is

$$F = [D, \bar{\partial}] = -\omega.$$

The importance of this structure was first emphasized in [BW].

From this data, one can construct the *scalar potential* V. It is the following function on M ([WB] formula 21.22; p. 169):

$$V = e^K \left(g^{i\bar{j}}(D_i W)(\bar{D}_{\bar{j}} W^*) - 3|W|^2 \right) \tag{18}$$

where $W^*(\bar{z})$ is the complex conjugate section and $\bar{D}_{\bar{j}} W^* = \bar{\partial}_{\bar{j}} W^* + (\bar{\partial}_{\bar{j}} K) W^*$.

The basic physics of the scalar potential is the following. While the fields z^i are functions on four-dimensional space-time, in a state of minimum energy (a ground state or vacuum) they will take constant values, at which the potential energy function $V(z)$ is a local minimum. In a given compactification and its corresponding supergravity theory T, there could be one, several or no such minima. The case of multiple minima is rather analogous to the familiar phenomenon of "phases of matter" such as solid, liquid, and gas, which have different expectation values for position-independent "fields" such as the local density, pressure, and so forth, which could be determined by minimizing a free energy. All physical predictions depend on the choice of minimum, and a first step to understanding the consequences of this is to know how many minima there are.

Whereas in general, the scalar potential in a field theory can be an arbitrary real function. in supergravity it must take the form (18), so this is a key formula in the physics of supersymmetry. Some of its features admit a more conceptual explanation. For example, the important fact that it is sesquilinear in W with signature $(n, 1)$, and is thus not positive definite, is the expected generalization of the familiar statement that a supersymmetric Hamiltonian is a sum of squares, to a theory containing gravity.

We define a *vacuum* to be a critical point $p \in \mathcal{C}$ of V. The vacua are further distinguished as follows:

- A *supersymmetric* vacuum is one in which the covariant gradient $D_i W = 0$. This can easily be seen to imply $V' = 0$, but the converse is not true.
- A *non-supersymmetric* vacuum is a critical point $V' = 0$, at which $D_i W \neq 0$. The norm of the gradient,

$$M_{susy}^4 \equiv e^K g^{i\bar{j}} D_i W D_{\bar{j}} W^*, \tag{19}$$

  is then referred to as the *scale of supersymmetry breaking*.
- The value of V at a critical point is the *cosmological constant* Λ of that vacuum. These are divided into $\Lambda = 0$, the Minkowski vacua, $\Lambda > 0$, the *de Sitter* (or dS) vacua, and $\Lambda < 0$, the *Anti-de Sitter* (or AdS) vacua. It is easy to see that supersymmetric vacua can only be Minkowski or AdS. The *Minkowski vacua* are simultaneous solutions of $D_i W = W = 0$; in this case $D_i W = \partial_i W$ and the existence of such vacua is independent of the Kähler potential. On the other hand, this is an overdetermined set of equations, so generic superpotentials do not have supersymmetric Minkowski vacua.

For our purposes, a *metastable* vacuum will be one for which the Hessian V'' is non-negative definite. Physically, this is required so that small fluctuations of the fields will not

grow exponentially.[1] We use the term metastable rather than stable, as such vacua have other potential instabilities (tunnelling) which we mention below.

We finally make a few comments about units. As with general relativity, in supergravity it is natural to work in "Planck units," in which the Planck scale, $M_P = 10^{19}$GeV in conventional units, is set to 1. If one knows the dimensions of a given quantity, it is easy to restore these factors. The fields z^i conventionally have dimension $[M_P]$ (this is chosen to make the action $\int |\partial z|^2$ dimensionless). The scalar potential V and the cosmological constant conventionally have dimension $[M_P^4]$, while the superpotential W has dimension $[M_P^3]$.

2.1. **An example from string theory.** Let us describe a simple example of an effective supergravity theory, which is known to arise from string theory, following the work of Giddings, Kachru and Polchinski [GKP]. Further details can be found in [AD].

One starts with the IIb superstring theory, and takes the $9+1$ space-time dimensions to be topologically $\mathbb{R}^{3,1} \times X$, where $\mathbb{R}^{3,1}$ is four-dimensional Minkowski space-time, and X is a Calabi-Yau manifold, a three complex dimensional compact Kähler manifold with zero first Chern class. It can be shown that $\dim H^{3,0}(X, \mathbb{C}) = 1$ and that the holomorphic three-form Ω is nowhere vanishing on X. By Yau's theorem, X admits a Ricci flat metric, so this space-time solves Einstein's equations.

Furthermore, the moduli space of Ricci flat metrics is isomorphic to the moduli space of complex structures on X, times a complexified Kähler cone. After compactification, this moduli space forms a factor in the supergravity configuration space M, and each point in M is a possible compactification. The Kähler metric is simply the Weil-Peterson metric on the moduli space (the natural metric on the space of metrics).

There is a natural line bundle $\mathcal{L}$ associated to the Kähler metric. As a bundle over complex structure moduli space, it has a simple geometric description: it is the Hodge line bundle $H^{3,0}(X, \mathbb{C}) \to M$ in which the holomorphic three-form Ω takes values. For more about the geometry associated to this situation, see [St, Fr2].

IIb superstring theory contains one more complex scalar field, the so-called "dilaton-axion". It parameterizes another factor in M, which is the upper half plane with the constant negative curvature metric. Though approximate, it is standard to take the metric on M to be a direct product of this metric, with the Weil-Peterson metric.

A simple example of a section of $\mathcal{L}$ is a period of Ω. The superpotentials are the following linear combinations of periods:

$$W = \int_X \Omega \wedge \left(F^{(1)} + \tau F^{(2)} \right), \tag{20}$$

where τ is the dilaton-axion and $F^{(1)}$ and $F^{(2)}$ are independently chosen elements of $H^3(M, \mathbb{Z})$.

This superpotential describes the contribution to the effective potential due to a "gauge field strength" or "flux" F. As a simple indication of this, we note that the formula (18) implies that V is quadratic in F, as is true for the energy of a magnetic field in Maxwell's theory, and as is true in supergravity. The standard argument for this superpotential [GKP] proceeds as follows. First, one can show that the critical points $DW = 0$ are points in moduli space at which the form $F^{(1)} + \tau F^{(2)}$ is purely in $H^{2,1}(X, \mathbb{C}) \oplus H^{0,3}(X, \mathbb{C})$. Second,

[1] This is evident in a Minkowski vacuum, but not literally true in a supersymmetric AdS vacuum; however this condition is still interesting in the latter context as it is the condition for stability when supersymmetry is broken by other (D term) effects.

this condition can be shown to imply that we are at a supersymmetric vacuum. Finally, (20) is the unique superpotential with these properties.

Thus, in this class of compactifications, we obtain a family of superpotentials, each of which is a linear combination of a finite basis of sections, the periods of the holomorphic three-form. To count vacua in this family of theories, we must count all the critical points of all of these sections which are allowed physically. This could be done by finding the expected number of critical points of a random section from this class, taken from an appropriate distribution, and multiplying by the number of distinct sections. Thus we have reformulated the physical problem as a problem in the statistics of holomorphic sections.

In fact, there is a physically well motivated choice for the ensemble of these sections, which we discuss in detail in [AD] and will return to in future work. It consists of the superpotentials (20) satisfying the constraint

$$\int_X F^{(1)} \wedge F^{(2)} = L \tag{21}$$

for some $L \in \mathbb{Z}$. Each superpotential is taken with weight 1, to obtain the total number of critical points.

The condition (21) sets the overall scale of W, and is analogous to the "spherical ensemble" of [BSZ1], of sections with a coefficient vector of unit length. While the coefficients here must be integers, in the limit $L \to \infty$, one expects the sections to be uniformly distributed on the constraint surface, and thus it should be a good approximation to neglect the quantization condition; the spherical or Gaussian ensembles should provide the large L asymptotics for the numbers of physical vacua.

There are further subtleties in making all of this precise, which we hope to return to in future work. For one thing, the quadratic form in (21) is indefinite. The reason this still defines a "spherical ensemble" is a slightly subtle argument which shows that upon restriction to the subspace of sections with a critical point at a chosen point in M, the form is positive definite.

In any case, this discussion should convince the reader that the problem of finding critical points of Gaussian random sections, is remarkably close to actual problems arising in string theory.

2.2. **Further physical questions about vacua.** Suppose we could find the vacua of the theories we just described, or of other compactifications of string/M theory: what physical questions would we like to answer? Let us discuss questions which can be answered with the data (M, K, W).

The most basic question is to count the supersymmetric and nonsupersymmetric vacua, or just the metastable ones. The simplest number to obtain is the "supergravity index," which counts critical points with a weight ± 1 as follows: if $W = 0$ at the critical point, the weight is $+1$, while if $W \neq 0$, the weight is the Morse index. This number is topological for M compact and nonsingular, and more generally is the integral of a topological density. Thus, it would be useful to obtain estimates for the other vacuum counts, in terms of this index, possibly under conditions such as bounds on curvature and its derivatives.

Besides counting the vacua, we might try to get a picture of their distribution in the configuration space M, by defining a measure whose integral over a region $R \subset M$ counts vacua within that region. To be precise, denote the candidate supergravity theories as T_a,

and within each of these, denote the critical points as z_i; the vacuum distribution is then

$$d\gamma[z_i] = \sum_{T_a} \sum_i \delta_{z_i}$$

Other distributions over vacua can be defined similarly. Let A_a be a function on M in a given theory T_a, for example the cosmological constant or supersymmetry breaking scale. We then define its distribution as

$$d\gamma[A] = \sum_{T_a} \sum_i \delta_{A_a(z_i)}.$$

A basic question about the supersymmetric vacua, is their distribution of cosmological constants $\Lambda = -3e^K|W|^2$, and especially the distribution near zero. It would be particularly interesting to find the distribution for flux superpotentials with integer coefficients.

For nonsupersymmetric vacua, one would like the joint distribution of cosmological constant Λ and supersymmetry breaking scale M_{susy}, ideally just for the metastable vacua.

Finally, one would like to consider more complete definitions of stability. In particular, a vacuum with $\Lambda = \Lambda_1 > 0$ can tunnel or decay to another vacuum with cosmological constant Λ_2 satisfying $\Lambda_1 > \Lambda_2 \geq 0$, at a rate roughly given by

$$\exp - \int dz \sqrt{V(z) - g_{i\bar{j}}(z)\dot{z}^i \dot{\bar{z}}^{\bar{j}}},$$

i.e. the exponential of an action, integrated along an action-minimizing trajectory between the two vacua. This formula is somewhat simplified, and more precise treatments can be found in [CDL, Ba, KKLT], but serves to illustrate the problem.

The total decay rate for a vacuum, is then the sum of this rate (and, possibly the rate for other decay processes), over all candidate target vacua. This consideration leads to a constraint which the vacuum describing our universe must satisfy: its decay rate should be smaller (hopefully, far smaller) than the inverse of the known time since the Big Bang, about 10^{10} years. Translated into Planck units, this is about 10^{-60}. Now in cases studied so far [KKLT], the decay rate to any single target vacuum is far smaller than this, around 10^{-100}, but it is conceivable that for M of high dimension, summing the rate over a large number of targets would lead to an interesting constraint.

3. Critical points of holomorphic sections

We begin the mathematical discussion with the definition of critical points of a holomorphic section $s \in H^0(M, L)$ relative to a connection ∇ on L. We recall that a smooth connection is a linear map

$$\nabla : \mathcal{C}^\infty(M, L) \to \mathcal{C}^\infty(M, L \otimes T^*)$$

satisfying $\nabla f s = df \otimes s + f\nabla s$ for $f \in \mathcal{C}^\infty(M)$. Choosing a local frame e_L of the line bundle L, we let

$$K(z) = -\log |e_L(z)|_h^2 \; . \tag{22}$$

The Chern connection $\nabla = \nabla_h$ is given by

$$\nabla(f\, e_L) = (df - f\partial K) \otimes e_L \; , \tag{23}$$

i.e., the connection 1-form (with respect to e_L) is $-\partial K$. We denote the curvature of h by

$$\Theta_h = -d\partial K = \partial\bar{\partial} K \; . \tag{24}$$

(Thus, a positive line bundle (L,h) induces the Kähler form $\omega = \frac{i}{2}\Theta_h = \frac{i}{2}\partial\bar\partial K$ with *Kähler potential* K.) By (23), $\nabla'' s = 0$ for any holomorphic section s where $\nabla = \nabla' + \nabla''$ is the splitting of the connection into its $L \otimes T^{*1,0}$, resp. $L \otimes T^{*0,1}$ parts.

Definition 3.1. *Let $(L,h) \to M$ be a holomorphic line bundle over a complex manifold, equipped with its Chern connection $\nabla = \nabla_h$. A critical point of a holomorphic section $s \in H^0(M,L)$ with respect to ∇ is defined to be a point $z \in M$ where $\nabla s(z) = 0$, or equivalently $\nabla' s(z) = 0$. We denote the set of critical points of s by $Crit^\nabla(s)$.*

It is important to understand that the set of critical points $Crit^\nabla(s)$ of s, and even its number $\# Crit^\nabla(s)$, depends on $\nabla = \nabla_h$ (or equivalently on the metric h). According to (23), the critical point condition in the local frame, $s = f e_L$, reads:

$$\partial f = f \partial K \iff \partial \log f = \partial K \ . \tag{25}$$

As mentioned in the introduction, this is a real $\mathcal{C}^\infty$ equation, not a holomorphic one since $\nabla s \in \mathcal{C}^\infty(M, L \otimes T^{*1,0})$ is a smooth but not holomorphic section and consequently does not always have positive intersection numbers with the zero section. Heuristically, the number of critical points reflects the degrees of both f and of K and the expected number of critical points should be large if the 'degree' of K is large.

An essentially equivalent definition in the case of a Chern connection is to define a critical point as a point w where

$$d|s(w)|_h^2 = 0. \tag{26}$$

Since

$$d|s(w)|_h^2 = 0 \iff 0 = \partial |s(w)|_h^2 = h_w(\nabla' s(w), s(w))$$

it follows that (26) is equivalent to $\nabla' s(w) = 0$ as long as $s(w) \neq 0$. So the critical point condition (26) gives the union of the zeros and critical points of the section s. Another essentially equivalent critical point equation which puts the zero set of s at $-\infty$ is

$$d \log |s(w)|_h^2 = 0. \tag{27}$$

This is the equation studied by Bott [Bo] in his Morse-theoretic proof of the Lefschetz hyperplane theorem, which is based on the observation that the Morse index of any such critical point is at least m. We shall use this observation to study the Morse index density in §7, where we note that the critical point theory of holomorphic sections at non-singular critical points is truly just the real Morse theory of the function $\log |s(z)|_h^2$.

We also note that the classical notion (cf. [AGV, Mi]) of critical point of a holomorphic function $f(z_1, \dots, z_m)$ on $\mathbb{C}^m$, i.e. a point w where

$$\frac{\partial f}{\partial z_1}(w) = \cdots \frac{\partial f}{\partial z_m}(w) = 0 \tag{28}$$

can be viewed as a connection critical point equation in the sense of Definition 3.1 but with a *meromorphic connection* rather than smooth Chern connection. That is, the derivatives $\frac{\partial f}{\partial z_j}$ on $\mathbb{C}^m$ define a meromorphic connection on the line bundles $\mathcal{O}(N) \to \mathbb{CP}^m$ with poles at infinity. Unlike the case of smooth connections, the critical point theory with respect to meromorphic connections is entirely a holomorphic theory. The critical points of a generic section in the sense of (28) all have topological index $+1$, and hence the number of critical points is a topological quantity depending on the polar variety of the meromorphic connection

and the Chern classes of M and L. This is in contrast to the case of a smooth connection, where the critical points of a generic section may have topological index -1 as well as $+1$, and their number depends on the section. As mentioned in the introduction (in the case of curves), the average number of critical points in the sense of Definition 3.1 is greater than the almost sure number in the classical sense.

The theory of critical points of holomorphic functions (cf. [AGV, Mi]) is concerned with the singularities of the hypersurface $f(z) = f(z_0)$ at a critical point z_0. The function $g(z) = f(z) - f(z_0)$ has a singular point at z_0, i.e. $g(z_0) = \nabla g(z_0) = 0$. The same notion of singular point applies to Definition 3.1 for holomorphic sections. We note that generic holomorphic sections and generic polynomials have no singular points. Those which do form the discriminant locus $\mathcal{D} \subset H^0(M, L)$. In physics terminology, singular points are known as Minkowski vacua. the statistics of singular points are quite different from those of critical points, and in particular $\mathcal{D}$ is a nonlinear subvariety of $H^0(M, L)$ and does not carry Gaussian measures.

3.1. **Hessians at a critical point.** There are three versions of the Hessian of s at a critical point which play a role in this paper. In this section, we define them and explain the relations between them.

The first version of the Hessian of s is

$$D\nabla s(z_0) \in (T^{*2,0} \oplus T^{*1,1}) \otimes L, \quad (\nabla s(z_0) = 0) \,, \tag{29}$$

Here, D is an auxiliary connection on $T^*M \otimes L$. As is well known, $D\nabla s(z_0)$ at a critical point is independent of the choice of D. This Hessian will be part of the jet map defined in (48).

The second version is the 'vertical part' $D^v\nabla s$ of the derivative of the section $\nabla s : M \to T^{*1,0} \otimes L$ with respect to a connection D on $T^{*1,0} \otimes L$. For lack of a standard term, we refer to it as the *complex Hessian* of s. This complex Hessian is the Hessian whose determinant appears in the statement of Theorem 1. It is defined as follows: From an invariant point of view, the connection gradient ∇s defines a section

$$\nabla s : M \to T^{*1,0} \otimes L. \tag{30}$$

We define $D^v\nabla s$ to be the vertical part of the derivative of (30) with respect to D. At a critical point $D^v\nabla s(z)$ is independent of the choice of the connection D. (The full derivative of ∇s maps TM to $T(T^{*1,0} \otimes L)$, which has real dimension $4m$, while $D^v\nabla s$ maps TM to the vertical tangent space $T^v(T^{*1,0} \otimes L) \approx T^{*1,0} \otimes L$.)

To define and compute the various Hessians, we introduce local coordinates and an adapted frame in the following sense:

Definition 3.2. *Let ∇ be the Chern connection on a Hermitian holomorphic line bundle $(L, h) \to M$. Let e_L be a local frame (non-vanishing holomorphic section) of L in a neighborhood of $z_0 \in M$, and let K be the local curvature potential given by* (22)*. We say that e_L is adapted to ∇ to order k at z_0 if all pure holomorphic derivatives of K of order $\leq k$ vanish at z_0 (and thus the pure anti-holomorphic derivatives also vanish). In particular, the connection form vanishes at z_0.*

We then write

$$\nabla s = \sum v_j \, dz_j \otimes e_L \,, \qquad v_j = \frac{\partial f}{\partial z_j} - f \frac{\partial K}{\partial z_j} \,. \tag{31}$$

We fix a point $z_0 \in M$ and choose an adapted local frame (of order 2) at z_0 as well as local normal holomorphic coordinates $z_1, \dots, z_m$ at z_0 (i.e., the connection form on T_M also vanishes at z_0 in these coordinates).

We then define linear functionals H'_{jq}, H''_{jq} (depending on our choice of coordinates and frame) on the space $H^0(M, L)$ by:

$$D'\nabla' s(z_0) = \sum_{j,q} H'_{jq} dz_q \otimes dz_j \otimes e_L, \qquad D''\nabla' s(z_0) = \sum_{j,q} H''_{jq} d\bar{z}_q \otimes dz_j \otimes e_L \,. \tag{32}$$

To obtain formulas for the matrices $H' = \big(H'_{jq}\big)$, $H'' = \big(H''_{jq}\big)$, we recall from (22) that

$$|e_L(z)|_h^2 = e^{-K(z)} \,, \tag{33}$$

and thus for a section $s = f e_L \in H^0(M, L)$, we have by (23):

$$\nabla s = \sum_{j=1}^m \left(\frac{\partial f}{\partial z_j} - f \frac{\partial K}{\partial z_j} \right) dz_j \otimes e_L = \sum_{j=1}^m e^K \frac{\partial}{\partial z_j} \left(e^{-K} f \right) dz_j \otimes e_L \,. \tag{34}$$

Differentiating (34), we then obtain:

$$H'_{jq} = \frac{\partial^2 f}{\partial z_j \partial z_q}(z_0) \,, \tag{35}$$

$$H''_{jq} = -f \frac{\partial^2 K}{\partial z_j \partial \bar{z}_q}\bigg|_{z_0} = -f(z_0)\Theta_{jq} \,, \quad \Theta_h(z_0) = \sum_{j,q} \Theta_{jq} dz_j \wedge d\bar{z}_q \,. \tag{36}$$

Thus, the standard Hessian $D\nabla s$ (see (29)) is given in our adapted coordinates and normal frame by the $m \times 2m$ matrix $\big(H' \ H''\big)$, where H' is a (complex-valued) symmetric matrix, and $H'' = -f(z_0)\Theta$, where Θ is the curvature matrix $\big(\Theta_{jq}\big)$.

To describe the complex Hessian $D^v\nabla s$, we begin by writing $z_q = x_q + iy_q$ and $v_j = \sigma_j + i\tau_j$ so that the real Jacobian matrix (at z_0) of ∇s with respect to the variables σ_j, τ_j and x_q, y_q and the local frame e_L is

$$\begin{pmatrix} \left(\frac{\partial \sigma_j}{\partial x_q}\right) & \left(\frac{\partial \sigma_j}{\partial y_q}\right) \\ \left(\frac{\partial \tau_j}{\partial x_q}\right) & \left(\frac{\partial \tau_j}{\partial y_q}\right) \end{pmatrix} \,. \tag{37}$$

But if we instead compute the Jacobian of ∇s with respect to the variables $v_j, \bar{v}_j$ and $z_q, \bar{z}_q$, we obtain the matrix

$$H^c := \begin{pmatrix} \left(\frac{\partial v_j}{\partial z_q}\right) & \left(\frac{\partial v_j}{\partial \bar{z}_q}\right) \\ \left(\frac{\partial \bar{v}_j}{\partial z_q}\right) & \left(\frac{\partial \bar{v}_j}{\partial \bar{z}_q}\right) \end{pmatrix} = \begin{pmatrix} H' & H'' \\ \overline{H''} & \overline{H'} \end{pmatrix} = \begin{pmatrix} H' & -f(z_0)\Theta \\ \overline{-f(z_0)\Theta} & \overline{H'} \end{pmatrix} \,. \tag{38}$$

Thus the complex Hessian is represented by the matrix H^c.

In invariant terms, at a critical point $\nabla s(z_0) = 0$, we may express $D^v\nabla s(z_0)$ as the matrix

$$D^v\nabla s(z_0) = \begin{pmatrix} \mathrm{Hess}_{hol}(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k})s(z) & \Theta(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_k})s(z) \\ & \\ \overline{\Theta(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_k})s(z)} & \overline{\mathrm{Hess}_{hol}(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k})s(z))} \end{pmatrix} . \tag{39}$$

relative to a basis of coordinate vector fields of local holomorphic coordinates. Here, the 'holomorphic Hessian' Hess_{hol} of s at a critical point is given by

$$\mathrm{Hess}_{hol}(Z,W)s = \nabla_{\widetilde{Z}}\nabla_{\widetilde{W}}s(z_0) \qquad Z,W \in T^{1,0}_{z_0}\,, \tag{40}$$

where $\widetilde{Z},\widetilde{W}$ are local holomorphic vector fields taking the values Z,W, respectively, at z_0. Indeed, (40) is clearly independent of the choice of $\widetilde{Z}$. Since the curvature Θ is of type $(1,1)$,

$$(\nabla_{\widetilde{Z}}\nabla_{\widetilde{W}}s - \nabla_{\widetilde{W}}\nabla_{\widetilde{Z}}s)(z_0) = \left(\nabla_{\widetilde{Z}}\nabla_{\widetilde{W}} - \nabla_{\widetilde{W}}\nabla_{\widetilde{Z}} - \nabla_{[\widetilde{Z},\widetilde{W}]}\right)s\,(z_0) = \Theta(Z,W)s(z_0) = 0\,,$$

it follows that $\mathrm{Hess}_{hol}(Z,W) = \mathrm{Hess}_{hol}(W,Z)$, which is also independent of the choice of $\widetilde{W}$.

The off-diagonal terms are the mixed' Hessian, given by

$$\mathrm{Hess}_{mixed}(Z,\overline{W})(s) = \nabla_{\overline{W}}\nabla_Z s(z_0).$$

Since

$$\Theta(Z,\overline{W})s(z_0) = \left(\nabla_Z\nabla_{\overline{W}} - \nabla_{\overline{W}}\nabla_Z - \nabla_{[Z,\overline{W}]}\right)s\,(z_0) = -\nabla_{\overline{W}}\nabla_Z s(z_0)$$

(here we dropped the $\widetilde{\ }$), the mixed Hessian equals

$$\mathrm{Hess}_{mixed}(Z,\overline{W})(s) = -\Theta(Z,\overline{W})s(z_0).$$

The third version is the usual Hessian of $\log|s|^2_h$ at a critical point. This version will be important in our discussion of Morse indices in §7. With respect to the basis $\{dz_j, d\bar{z}_j\}$, it is given at a critical point z_0 by

$$\begin{aligned}\begin{pmatrix} \left(\frac{\partial^2}{\partial z_j\partial z_q}\log|s|^2_h\right) & \left(\frac{\partial^2}{\partial z_j\partial \bar{z}_q}\log|s|^2_h\right) \\ \left(\frac{\partial^2}{\partial \bar{z}_j\partial z_q}\log|s|^2_h\right) & \left(\frac{\partial^2}{\partial \bar{z}_j\partial \bar{z}_q}\log|s|^2_h\right)\end{pmatrix} &= \begin{pmatrix} \left(\frac{1}{f}\frac{\partial^2 f}{\partial z_j\partial z_q}\right) & \left(-\frac{\partial^2 K}{\partial z_j\partial \bar{z}_q}\right) \\ \left(-\frac{\partial^2 K}{\partial \bar{z}_j\partial z_q}\right) & \left(\frac{1}{\bar{f}}\frac{\partial^2 \bar{f}}{\partial \bar{z}_j\partial \bar{z}_q}\right)\end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{f(z_0)}H' & -\Theta \\ -\overline{\Theta} & \frac{1}{\bar{f}(z_0)}\overline{H'}\end{pmatrix} = \begin{pmatrix} f(z_0)^{-1} & 0 \\ 0 & \bar{f}(z_0)^{-1}\end{pmatrix} H^c\,.\end{aligned} \tag{41}$$

Note that the matrix (41) is not Hermitian. In §7, we use a Hermitian version of (41) obtained by conjugating the real Hessian of $\log|s|^2_h$ by a unitary matrix; the resulting Hermitian matrix (108) contains the entries of (41), re-arranged and with constant factors.

4. A density formula for zeros

We now begin the study of the statistics of critical points of random sections $s \in \mathcal{S} \subset H^0(M,L)$ with respect to a complex Gaussian measure γ. We recall that a complex Gaussian measure is induced by a choice of Hermitian inner product $\langle,\rangle$ on $\mathcal{S} \subset H^0(M,L)$:

Definition 4.1. *We define the Gaussian measure associated to* $(\mathcal{S},\langle,\rangle)$ *by*

$$d\gamma(s) = \frac{1}{\pi^d}e^{-\|c\|^2}dc\,, \qquad s = \sum_{j=1}^d c_j e_j, \tag{42}$$

where dc is Lebesgue measure and $\{e_j\}$ is an orthonormal basis for $\mathcal{S}$ relative to $\langle,\rangle$.

We denote the expected value of a random variable X on $\mathcal{S}$ with respect to γ by $\mathbf{E}_\gamma X$ or simply by $\mathbf{E}X$ when $(\mathcal{S},\gamma)$ are understood. We recall from (2)–(3) that the expected distribution of critical points of $s\in\mathcal{S}\subset H^0(M,L)$ with respect to $(\mathcal{S},\gamma,\nabla)$ is the measure $\mathbf{K}^{\rm crit}_{\mathcal{S},\gamma,\nabla}=\mathbf{E}\,_\gamma C^\nabla_s$ on M, where

$$C^\nabla_s=\sum_{z\in Crit^\nabla(s)}\delta_z\ ,$$

where δ_z is the Dirac point mass at z. Thus,

$$\left(\mathbf{K}^{\rm crit}_{\mathcal{S},\gamma,\nabla},\varphi\right)=\int_{\mathcal{S}}\left[\sum_{z:\nabla s(z)=0}\varphi(z)\right]d\gamma(s). \tag{43}$$

4.1. **Covariance kernel.** The crucial invariant of a Gaussian measure is its covariance or two-point kernel:

Definition 4.2. *The two-point kernel of a Gaussian measure γ defined by $(\mathcal{S},\langle,\rangle)$ is defined by*

$$\Pi_{\mathcal{S}}(z,w)=\mathbf{E}_{\mathcal{S}}(s(z)\otimes\overline{s}(w))\in L_z\otimes\overline{L_w}.$$

Here $\overline{L}$ denotes the complex conjugate of the line bundle L (characterized by the existence of a conjugate linear bijection $L\stackrel{\approx}{\to}\overline{L},\ v\mapsto\bar v$). As is well-known and easy to see, $\Pi_{\mathcal{S}}$ can be written in the form

$$\Pi_{\mathcal{S}}(z,w)=\sum_{j=1}^n s_j(z)\otimes\overline{s_j(w)}\ ,$$

where $\{s_1,\dots s_n\}$ is an orthonormal basis for $\mathcal{S}$ with respect to the inner product $\langle,\rangle$ associated to the Gaussian measure γ. Indeed,

$$\mathbf{E}\left(s(z)\otimes\overline{s(w)}\right)=\mathbf{E}\left(\sum_{j,k=1}^n c_j\overline{c_k}\,s_j(z)\otimes\overline{s_k(w)}\right)=\sum_{j=1}^n s_j(z)\otimes\overline{s_j(w)}\ , \tag{44}$$

since the c_j are independent complex (Gaussian) random variables of variance 1.

In the case of a Hermitian line bundle, the two point kernel of the Hermitian Gaussian measure is the Szegö kernel of (L,h), i.e. the orthogonal projection $\Pi_{\mathcal{S},h,V}:\mathcal{L}^2(M,L)\to\mathcal{S}$ with respect to the inner product (10).

4.2. **Expected density of random discrete zeros.** The expected density of critical points may be regarded as the expected density of zeros of random sections in the subspace $\nabla H^0(M,L)\subset\mathcal{C}^\infty(M,T^{*1,0}\otimes L)$. In this section, we prove a general formula (Theorem 4.4) for the density of zeros of random sections which applies to this subspace and which will be used to prove Theorem 1. It may be derived from the rather general and abstract Theorem 4.2 of [BSZ2]. However, that theorem gives the n-point correlation of zeros of several random sections in all codimensions, while here we consider only the density (or "1-point correlation") in the full codimension case where the zeros are discrete. This is both simpler than the general setting in [BSZ2] and also involves some special features not quite covered there. To make the paper more self-contained, we give a derivation from scratch of the density formula for discrete zeros that arises from [BSZ2, Theorem 4.2]. In §5.4, we give an alternate approach to the proof which is closer to [AD].

The general set-up in [BSZ2] involves 1-jets of sections of a real vector bundle V over a smooth manifold M. (We shall later apply our formula to the case where $V = T^{*1,0} \otimes L$ is complex, but the sections $\nabla s \in \mathcal{S}$ are not holomorphic.) For simplicity of exposition, we will endow V with a connection ∇ and an inner product h, and we will endow TM with a Riemannian metric and a volume form $d\mathrm{Vol}_M$. The result of Theorem 4.4 below is independent of these choices of connection and metric.

Let $\mathcal{S} \subset \mathcal{C}^\infty(M, V)$ be a finite-dimensional subspace of smooth sections and consider the *jet maps*

$$J^1_z : \mathcal{S} \to J^1(M, V)_z \ , \qquad z \in M \ ,$$

where $J^1(M, V)$ denotes the vector bundle of 1-jets of sections of V, and $J^1_z(s)$ is the 1-jet at $z \in M$ of a section $s \in \mathcal{S}$. Recall that we have the canonical vector bundle exact sequence

$$0 \to T^*_M \otimes V \to J^1(M, V) \xrightarrow{\epsilon} V \to 0 \ , \tag{45}$$

where ϵ is the evaluation map.

The connection ∇ on V gives a splitting of (45),

$$(\varepsilon, \nabla) : J^1(M, V) \xrightarrow{\approx} V \oplus (T^*_M \otimes V) \ , \qquad J^1_z(s) \mapsto (s(z), \nabla s(z)) \ . \tag{46}$$

We shall identify $V \oplus (T^*_M \otimes V)$ with the space $J^1(M, V)$ of 1-jets via (46). Given a Gaussian measure γ on $\mathcal{S}$ and a point $z \in M$, we consider the pushforward measure

$$\mathbf{D}_z := (J^1_z)_* \gamma \ , \tag{47}$$

which is called the *joint probability distribution* of γ. Since the jet map J^1_z is Gaussian, the joint probability distribution $\mathbf{D}_z$ is likewise Gaussian.

In the application we have in mind, $V = T^{*1,0} \otimes L$, $\mathcal{S} = \nabla H^0(M, L)$ and

$$J^1_z : \nabla H^0(M, L) \to J^1(M, T^{*1,0} \otimes L) \approx \left(T^{*1,0} \oplus [T^{*1,0} \otimes T^{*1,0}] \oplus [T^{*0,1} \otimes T^{*1,0}]\right)_z \otimes L_z. \tag{48}$$

A complication arises (when $\dim M > 1$) in that the range of J^1_z is a proper subspace of $J^1(M, V)_z$. Indeed, in terms of normal coordinates, $J^1(M, T^{*1,0} \otimes L)$ can be identified with the space of triples (v, H', H''), where $v \in \mathbb{C}^m$ and H', H'' are complex $m \times m$ matrices, while the range of J^1_z consists only of those triples where H' is a complex symmetric matrix and $H'' = x\,\Theta$, $x \in \mathbb{C}$ (see (35)–(36)). Then $\mathbf{D}_z$ becomes a singular Gaussian measure on $J^1(M, V)$. The results of [BSZ1] include singular measures, but it is simpler to apply the results in a way which is better adapted to the subspace situation.

Hence, returning to our general setup, we assume that the jet map has the following spanning property:

Definition 4.3. *Let $\mathcal{S}$ be a linear space of sections of a $\mathcal{C}^\infty$ vector bundle $V \to M$ and let $\mathcal{J}^1 : M \times \mathcal{S} \to J^1(M, V)$ be given by $\mathcal{J}^1(z, s) = J^1_z(s)$. We say that $\mathcal{S}$ has the spanning property with respect to a sub-bundle $W \subset T^*_M \otimes V$ if* $\mathrm{Image}\,\mathcal{J}^1$ *is a sub-bundle of $J^1(M, V)$ and $\epsilon : \mathrm{Image}\,\mathcal{J}^1 \to V$ is surjective with kernel W; i.e.,*

$$0 \to W \to \mathrm{Image}\,\mathcal{J}^1 \xrightarrow{\epsilon} V \to 0 \tag{49}$$

is an exact sequence of vector bundles.

The pushforward measure $\mathbf{D}_z$ of (47) is then a (nonsingular) Gaussian measure on $\mathrm{Image}\, J^1_z$. Making the identification $J^1(M, V) \approx V \oplus (T^*_N \otimes V)$ via (46), we have

$$\mathrm{Image}\, J^1_z \approx V_z \oplus W_z \ . \tag{50}$$

We then regard $\mathbf{D}_z$ as a Gaussian measure on $V_z \oplus W_z$, and we write

$$\mathbf{D}_z = D(x,\xi;z)\,dx\,d\xi \qquad (z \in M,\ x \in V_z,\ \xi \in W_z)\,, \tag{51}$$

where dx, $d\xi$ denote Lebesgue measure on V_z, W_z respectively (with respect to our Riemannian metric G on M and inner product h on V). We note that $D(x,\xi;z)$ depends on the choice of metrics, but of course $\mathbf{D}_z$ does not.

We now assume further that $\operatorname{rank} V = \dim M = k$, so that by the spanning property, the zero sets Z_s of sections $s \in \mathcal{S}$ are almost surely discrete. We shall denote by $|Z_s|$ the sum of delta functions at the zeros of s. The following theorem is a special case of Theorem 4.2 in [BSZ2]:

Theorem **4.4.** *Let $V \to M$ be a $\mathcal{C}^\infty$ real vector bundle over a $\mathcal{C}^\infty$ manifold of dimension $k = \operatorname{rank}(V)$, and let $\mathcal{S} \subset \mathcal{C}^\infty(M,V)$ be a finite-dimensional subspace with the spanning property* (49) *with respect to a subspace $W \subset T^*_M \otimes V$. Let γ be a Gaussian probability measure on $\mathcal{S}$. Then*

$$\mathbf{E}_\gamma |Z_s| = \mathcal{K}\, d\mathrm{Vol}_M\,, \quad \mathcal{K}(z) = \int_{W_z} D(0,\xi;z)\,\|\det \xi\|\,d\xi\,, \tag{52}$$

where $d\xi$ denotes Lebesgue measure with respect to the metric on $W_z \subset T^*_{M,z} \otimes V_z$, and where $D(0,\xi;z)$ is given by (47) *and* (51)*. (An explicit formula for $D(0,\xi;z)$ is given in (59).)*

The notation $\|\det \xi\|$ in (52) is defined as follows: a V-valued 1-form $\xi \in (T^*_M \otimes V)_z = \operatorname{Hom}(T_{M,z}, V_z)$ induces a $(\det V)$-valued k-form

$$\det \xi \in \operatorname{Hom}(\det T_{M,z}, \det V_z) = \left(\bigwedge\nolimits^k T^*_M \otimes \det V\right)_z\,.$$

Then $\|\det \xi\|$ is the norm on $\det T^*_{M,z} \otimes \det V_z$ induced from the metrics on M and V. To describe the norm explicitly, we write

$$\xi = \sum_{j=1}^k \xi_j \otimes e_j, \qquad \xi_j \in T^*_z,$$

where $\{e_1,\dots,e_k\}$ is an orthonormal basis for V_z. Then

$$\|\det \xi\| = \|\xi_1 \wedge \cdots \wedge \xi_k\| = \left|\frac{\xi_1 \wedge \cdots \wedge \xi_k}{d\mathrm{Vol}_M}\right|\,. \tag{53}$$

Remark: We note that $D(0,\xi;z)$ is independent of the choice of the connection ∇ (see [BSZ1, p. 371]). (It does depend on the choice of metric on V, but the reader can easily check that $D(0,\xi;z)\,\|\det \xi\|\,d\xi\, d\mathrm{Vol}_M(z)$ defines a measure on W that is independent of metrics and volume forms.)

4.2.1. *Zeros of sections of complex vector bundles.* Now let $V \to M$, $\mathcal{S} \subset \mathcal{C}^\infty(M,V)$ be as in Theorem 4.4, but let V be a complex vector bundle of rank k over $\mathbb{C}$. We suppose that $\dim M = 2k$ so that we have point zeros. We may apply Theorem 4.4, regarding $V \to M$ as a real vector bundle of rank $2k$.

Then (52) holds, but we must properly interpret $\|\det\xi\|$. To do this, we fix $z\in M$, and we pick an orthonormal basis $\{e_1,\dots,e_k\}$ of V_z over $\mathbb{C}$. We then regard V_z as a real vector bundle endowed with the inner product having orthonormal basis

$$\left\{\frac{1}{\sqrt{2}}e_1,\frac{i}{\sqrt{2}}e_1,\dots,\frac{1}{\sqrt{2}}e_k,\frac{i}{\sqrt{2}}e_k\right\}\ .$$

As before, for $\xi\in(T^*_M\otimes V)_z$, we write

$$\xi=\sum_{j=1}^k\xi_j\otimes e_j=\sum_{j=1}^k\left(\mathrm{Re}\xi_j\otimes e_j+\mathrm{Im}\xi_j\otimes ie_j\right),\qquad \xi_j\in T^*_z\otimes\mathbb{C}\ .$$

Thus we have

$$\|\det\xi\|=2^k\|\mathrm{Re}\xi_1\wedge\mathrm{Im}\xi_1\wedge\cdots\wedge\mathrm{Re}\xi_k\wedge\mathrm{Im}\xi_k\|=\|\xi_1\wedge\cdots\wedge\xi_k\wedge\bar\xi_1\wedge\cdots\wedge\bar\xi_k\|\ .\qquad(54)$$

4.2.2. *Proof of Theorem 4.4.* As mentioned above, the theorem is a special case of Theorem 4.2 in [BSZ2]. However, the proof in this case (which is based on the proof in [BSZ1]) is quite simple, so we present it here.

We can restrict to a neighborhood U of an arbitrary point $z_0\in M$. Since $\mathcal{S}$ spans V, we can choose U so that there exist sections $e_1,\dots,e_k\in\mathcal{S}$ that form a local frame for V over U. For a section $s\in\mathcal{S}$, we write $s(z)=\sum_{j=1}^k s_j(z)e_j(z)$ $(z\in U)$ and we let $\tilde s=(s_1,\dots,s_k):U\to\mathbb{R}^k$. Since $D(0,\xi;z)$ is independent of the connection, we can further assume that $\nabla|_U$ is the flat connection $\nabla s=\sum ds_j\otimes e_j$. Then

$$\|\det\nabla s\|=\sqrt{h}\|ds_1\wedge\cdots\wedge ds_k\|\,,\qquad(55)$$

where $h=\det\big(h(e_j,e_{j'})\big)$.

We let $\psi_\varepsilon\to\delta_0$ be an approximate identity on $\mathbb{R}^k$, and we write $c=(c_1,\dots,c_k)\in\mathbb{R}^k$, $dc=dc_1\cdots dc_k$. For a test function $\varphi\in\mathcal{D}(U)$, we have by (53) and (55),

$$\begin{aligned}\int_{\mathbb{R}^k}\psi_\varepsilon(c)\Big(|\tilde s^{-1}(c)|,\varphi\Big)\,dc &= \int_{\mathbb{R}^k}\psi_\varepsilon(c)\Big[\sum_{\tilde s(z)=c}\varphi(z)\Big]\,dc\\ &= \int_U(\psi_\varepsilon\circ\tilde s)\ \varphi\,|ds_1\wedge\cdots\wedge ds_k|\\ &= \int_U(\psi_\varepsilon\circ\tilde s)\ \varphi\,\|\det\nabla s\|\,h^{-1/2}\,d\mathrm{Vol}_M\ .\end{aligned}\qquad(56)$$

Integrating (56) over $\mathcal{S}$ and using (47), we obtain

$$\begin{aligned}\int_{\mathbb{R}^k}\psi_\varepsilon(c)\big(\mathbf{E}\,|\tilde s^{-1}(c)|,\varphi\big)\,dc &= \int_{\mathcal{S}}\int_U(\psi_\varepsilon\circ\tilde s)(z)\,\varphi(z)\,\|\det\nabla s\|_z\,h(z)^{-1/2}\,d\mathrm{Vol}_M(z)\,d\gamma(s)\\ &= \int_M\int_{W_z}\int_{\mathbb{R}^k}\psi_\varepsilon(c)\,\varphi(z)\,\|\det\xi\|\,D\left(\textstyle\sum c_je_j,\xi;z\right)dc\;d\xi\,d\mathrm{Vol}_M(z),\end{aligned}\qquad(57)$$

where the latter equality follows from the fact that

$$(J^1_z)_*(d\gamma)=D(x,\xi;z)\,dx\,d\xi=D\left(\textstyle\sum c_je_j,\xi;z\right)\ h(z)^{1/2}\,dc\,d\xi\ .$$

Letting $\varepsilon \to 0$ in (57), we obtain

$$\mathbf{E}\left(|\tilde{s}^{-1}(0)|, \varphi\right) = \int_M \int_{W_z} \varphi(z) \, \| \det \xi \| \, D\left(0, \xi; z\right) \, d\xi \, d\mathrm{Vol}_M(z) \ .$$

Recalling that $\tilde{s}^{-1}(0) = Z_s$, we then obtain (52). □

Remark: The proof of the analogous result for the case where $\operatorname{rank} V < \dim M$ follows the same argument. The only additional ingredient is Federer's co-area formula, which is used to obtain (56); see [BSZ2].

4.3. **Description of the joint probability distribution.** We again suppose that V is a complex vector bundle. Recall that the measure $\mathbf{D}_z$ is the pushforward of the Gaussian measure γ under the linear map J^1_z. Since the push-forward of a Gaussian measure under a linear map is Gaussian, $\mathbf{D}_z$ is a Gaussian measure on Image J^1_z. We now give a formula for $\mathbf{D}_z$ and more importantly, for the conditional Gaussian measure $\mathbf{D}^0_z$ that appears in our formula (52).

Let $z \in M$, and choose orthonormal bases $\{e_1, \ldots, e_k\}$, $\{w_1, \ldots, w_n\}$ of V_z, W_z, respectively. The Gaussian measure $\mathbf{D}_z$ can be written in the form

$$d\gamma_{\Delta(z)}(v, w) = \frac{1}{\pi^{k+n} \det \Delta(z)} \exp\left[- \left\langle \Delta(z)^{-1} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle \right] dx_1 \cdots dx_k \, dy_1 \cdots dy_n \ ,$$

$$v = \sum_{j=1}^{k} x_j e_j \ , \ w = \sum_{q=1}^{n} y_q w_q \ .$$

The covariance matrix $\Delta(z)$ is given in block form by

$$\Delta(z) = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}, \quad A = \left[\mathbf{E}\left(x_j \bar{x}_{j'}\right)\right], \ B = \left[\mathbf{E}\left(x_j \bar{y}_q\right)\right], \ C = \left[\mathbf{E}\left(y_q \bar{y}_{q'}\right)\right], \tag{58}$$

$$1 \leq j, j' \leq k, \ 1 \leq q, q' \leq n.$$

Using the formula for the inverse of a matrix in block form, we obtain

$$D(0, y; z) = \frac{1}{\pi^{k+n} \det A \det \Lambda} \exp\left(-\langle \Lambda^{-1} y, y \rangle \right) \ , \tag{59}$$

where $\Lambda = C - B^* A^{-1} B$ as in (7).

5. Density formulas: Proof of Theorem 1

We now prove Theorem 1 for the ensemble $(\mathcal{S}, \nabla, \gamma)$ by applying the zero-density formula (52)–(53) of Theorem 4.4 to the ensemble

$$\mathcal{S}' := \nabla \mathcal{S} \subset \mathcal{C}^\infty(M, T^*_M \otimes L) \ ,$$

endowed with the Gaussian probability measure on $\mathcal{S}'$ induced by γ. We assume that $\mathcal{S}$ has the following property:

Definition 5.1. *Let $(L, h) \to M$ be a Hermitian holomorphic line bundle. We say that $\mathcal{S} \subset H^0(M, L)$ has the 2-jet spanning property if the jet maps*

$$J^2_z : \mathcal{S} \to J^2_{\mathrm{Hol}}(M, L)_z$$

are surjective for all $z \in M$ (where $J^2_{\rm Hol}(M,L)$ denotes the vector bundle of 2-jets of holomorphic sections of L).

When $L \to M$ is a positive line bundle on a compact complex manifold M, the surjectivity of J^2_z always holds for $\mathcal{S} = H^0(M, L^N)$ when N is sufficiently large, as a well known consequence of the Kodaira Vanishing Theorem.

We begin with the following observation:

Lemma **5.2.** *Let $(L,h) \to M$ be a Hermitian holomorphic line bundle such that the Chern connection ∇ had nonvanishing curvature form Θ. Suppose that $\mathcal{S} \subset H^0(M,L)$ is a linear space of sections with the 2-jet spanning property. Then $\nabla\mathcal{S} \subset \mathcal{C}^\infty(M, T^{*1,0}_M \otimes L)$ has the spanning property with respect to*

$$W := (S^2 T^{*1,0}_M \oplus \mathbb{C}\widetilde{\Theta}) \otimes L \subset (T^{*1,0}_M \otimes T^{*1,0}_M \otimes L) \oplus (T^{*0,1}_M \otimes T^{*1,0}_M \otimes L) = T^*_M \otimes T^{*1,0}_M \otimes L\ ,$$

where $S^2 T^{*1,0}_M \subset T^{*1,0}_M \otimes T^{*1,0}_M$ denotes the symmetric tensors, and $\widetilde{\Theta}$ corresponds to Θ under the natural identification $T^{*1,1}_M \approx T^{*0,1}_M \otimes T^{*1,0}_M$.

Proof. We begin by describing the relevant random variables $x, v_j, H'_{jq}, H''_{jq}$ used to describe the jet map J^1_z. Let $z_0 \in M$ and choose normal coordinates $\{z_j\}$ and a special frame e_L adapted to ∇ at z_0. Recalling (31)–(32), we consider the linear functionals $x, v_j, H'_{jq}, H''_{jq}$ on the space $H^0(M,L)$ given by:

$$s(z_0) = x\, e_L, \qquad \nabla s(z_0) = \nabla' s(z_0) = \sum_{j=1}^m v_j dz_j \otimes e_L \tag{60}$$

$$D'\nabla' s(z_0) = \sum_{j,q} H'_{jq} dz_q \otimes dz_j \otimes e_L, \qquad D''\nabla' s(z_0) = \sum_{j,q} H''_{jq} d\bar z_q \otimes dz_j \otimes e_L\,. \tag{61}$$

The jet map in local coordinates, using the identification (48), is given by

$$J^1_{z_0} = (v_j, H'_{jq}, H''_{jq})\ . \tag{62}$$

The conclusion is an immediate consequence of the 2-jet spanning property of $\mathcal{S}$ and (35)–(36). □

We recall that the matrices $[H'_{jq}]$ and $[H''_{jq}]$ are the coordinate representation of the holomorphic Hessian and mixed Hessian described in §3.1, where it was observed that they form part of the vertical derivative matrix H^c of ∇s.

5.1. **Density formula and covariance kernel.** Following §4.3, we next compute the joint probability density using the coordinates $\{H'_{jq}\ (1 \le j \le q \le m),\ x\}$ with respect to the basis

$$\{dz_j \otimes dz_q \otimes e_L|_{z_0}\ (1 \le j \le q \le m),\ \Theta_h \otimes e_L|_{z_0}\}$$

of W_{z_0}. (Here, in order to obtain the result as a consequence of Theorem 4.4 on zero densities, we need to assume that the curvature form Θ_h does not vanish at z_0. However, in the general case, the formula follows directly from the argument in §4.2.2 using instead

the joint probability distribution $D(v, H', x; z_0)\, dv\, dH'\, dx$.) The joint probability density $D(v, H', x; z_0)$ is Gaussian with covariance matrix $\Delta(z_0)$ given by:

$$\Delta(z_0) = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}, \tag{63}$$

$$A = \left[\mathbf{E}\left(v_j \overline{v_{j'}}\right)\right], \tag{64}$$

$$B = \left[\mathbf{E}\left(v_j \overline{H'_{j'q'}}\right) \quad \mathbf{E}\left(v_j \bar{x}\right)\right], \tag{65}$$

$$C = \left[\begin{array}{cc} \mathbf{E}\left(H'_{jq}\overline{H'_{j'q'}}\right) & \mathbf{E}\left(H'_{jq}\bar{x}\right) \\ \mathbf{E}\left(x\overline{H'_{j'q'}}\right) & \mathbf{E}\left(|x|^2\right) \end{array}\right], \tag{66}$$

$$1 \le j \le m\,, 1 \le j \le q \le m\,, 1 \le j' \le q' \le m\,.$$

We now describe how $\Delta(z_0)$ is given in terms of the covariance kernel $\Pi_{\mathcal{S}}$ of $\mathcal{S}$ (cf. Definition 4.2). It is in fact simpler to use the local expression for the covariance kernel in a local frame (non-vanishing local holomorphic section). We fix a point $z_0 \in M$, and choose a frame e_L of L on a neighborhood $U \subset M$ of z_0. We write every section in the form $s = f\, e_L$.

Definition 5.3. *The local covariance kernel $F_{\mathcal{S}}(z, w) \in \mathcal{O}(U \times \overline{U})$ in a frame e_L of L is defined by*

$$\Pi_{\mathcal{S}}(z, z) = F_{\mathcal{S}}(z, z)\, e_L(z) \otimes \overline{e_L(z)}\ .$$

Equivalently,

$$F_{\mathcal{S}}(z_0, w_0) = \sum_j f_j(z_0)\overline{f}_j(w_0)$$

where $s_j = f_j e_L$ is an orthonormal basis of $(\mathcal{S}, \langle,\rangle)$.

We then have:

$$\mathbf{E}\left(|x|^2\right) = F_{\mathcal{S}}(z_0, z_0) = \sum |f_j(z_0)|^2. \tag{67}$$

We emphasize that both the random variable x of (60) and the formula (67) depend on the choice of frame e_L. It is convenient to introduce an invariant notation for the local covariance kernel in the frame e_L. We write (67) as

$$\mathbf{E}\left(|x|^2\right) = \frac{\Pi_{\mathcal{S}}(z_0, z_0)}{e_L(z_0) \otimes \overline{e_L}(z_0)} = \rho^{\rm diag}_{e_L(z_0)}\, \Pi_{\mathcal{S}}\ , \tag{68}$$

where $\rho^{\rm diag}$ denotes the restriction to the diagonal, and

$$\rho^{\rm diag}_v G = G(z_0, z_0)/(v \otimes \bar{v}) \in \mathbb{C}\ , \tag{69}$$

for $G(z_0, z_0) \in L_{z_0} \otimes \bar{L}_{z_0},\ v \in L_{z_0}$.

Differentiating (44), we obtain

$$\mathbf{E}\left(\nabla_{z_j} s(z) \otimes \overline{\nabla_{w_{j'}} s(w)}\right) = \nabla_{z_j} \nabla_{\bar{w}_{j'}} \Pi_{\mathcal{S}}(z, w)\ ,$$

where we write

$$\nabla' s = \sum_{j=1}^m dz_j \otimes \nabla_{z_j} s\,, \quad \nabla'' s = \sum_{j=1}^m d\bar{z}_j \otimes \nabla_{\bar{z}_j} s\ .$$

Hence,

$$\mathbf{E}\left(v_j\overline{v_{j'}}\right) = \rho^{\mathrm{diag}}_{e_L(z_0)} \nabla_{z_j}\nabla_{\bar w_{j'}}\Pi_{\mathcal{S}}\ .$$

Thus, after repeatedly differentiating (5.3), the matrices (64)–(66) can be expressed in terms of the covariance kernel and its covariant derivatives on the diagonal:

$$A = \left(\rho^{\mathrm{diag}}_{e_L(z_0)}\nabla_{z_j}\nabla_{\bar w_{j'}}\Pi_{\mathcal{S}}\right), \tag{70}$$

$$B = \left[\left(\rho^{\mathrm{diag}}_{e_L(z_0)}\nabla_{z_j}\nabla_{\bar w_{q'}}\nabla_{\bar w_{j'}}\Pi_{\mathcal{S}}\right) \quad \left(\rho^{\mathrm{diag}}_{e_L(z_0)}\nabla_{z_j}\Pi_{\mathcal{S}}\right)\right], \tag{71}$$

$$C = \left[\begin{array}{cc} \left(\rho^{\mathrm{diag}}_{e_L(z_0)}\nabla_{z_q}\nabla_{z_j}\nabla_{\bar w_{q'}}\nabla_{\bar w_{j'}}\Pi_{\mathcal{S}}\right) & \left(\rho^{\mathrm{diag}}_{e_L(z_0)}\nabla_{z_q}\nabla_{z_j}\Pi_{\mathcal{S}}\right) \\ \left(\rho^{\mathrm{diag}}_{e_L(z_0)}\nabla_{\bar w_{q'}}\nabla_{\bar w_{j'}}\Pi_{\mathcal{S}}\right) & \rho^{\mathrm{diag}}_{e_L(z_0)}\Pi_{\mathcal{S}} \end{array}\right], \tag{72}$$

$$1\le j\le m\,, 1\le j\le q\le m\,, 1\le j'\le q'\le m\,.$$

In the above, A, B, C are $m\times m$, $m\times n$, $n\times n$ matrices, respectively, where $n=\frac12(m^2+m+2)$.

We pause to obtain simple local formulas in an adapted frame and in normal coordinates for ∇. We first replace each covariant derivative by its local expression $\nabla_{z_j}=\frac{\partial}{\partial z_j}-\frac{\partial K}{\partial z_j}$ in the frame e_L and each $\Pi_{\mathcal{S}}$ can be replaced by its local expression $F_{\mathcal{S}}$. Thus,

$$\mathbf{E}\left(\nabla_{z_j}s(z)\otimes\overline{\nabla_{w_{j'}}s(w)}\right) = \left(\frac{\partial}{\partial z_j}-\frac{\partial K}{\partial z_j}\right)\left(\frac{\partial}{\partial \bar w_{j'}}-\frac{\partial K}{\partial \bar w_{j'}}\right)F_\gamma(z,w)|_{z=w}. \tag{73}$$

Similarly for higher covariant derivatives.

Thus we have,

$$A = \left(\left(\tfrac{\partial}{\partial z_j}-\tfrac{\partial K}{\partial z_j}\right)\left(\tfrac{\partial}{\partial \bar w_{j'}}-\tfrac{\partial K}{\partial \bar w_{j'}}\right)F_{\mathcal{S}}(z,w)|_{z=w}\right), \tag{74}$$

$$B = \left[\left(\left(\tfrac{\partial}{\partial z_j}-\tfrac{\partial K}{\partial z_j}\right)\left(\tfrac{\partial}{\partial \bar w_{q'}}-\tfrac{\partial K}{\partial \bar w_{q'}}\right)\left(\tfrac{\partial}{\partial w_{j'}}-\tfrac{\partial K}{\partial \bar w_{j'}}\right)F_{\mathcal{S}}|_{z=w}\right) \quad \left(\left(\tfrac{\partial}{\partial z_j}-\tfrac{\partial K}{\partial z_j}\right)F_{\mathcal{S}}|_{z=w}\right)\right], \tag{75}$$

$$C = \left[\begin{array}{cc} C' & \left(\left(\tfrac{\partial}{\partial z_j}-\tfrac{\partial K}{\partial z_j}\right)\left(\tfrac{\partial}{\partial z_q}-\tfrac{\partial K}{\partial z_q}\right)F_{\mathcal{S}}|_{z=w}\right) \\ \left(\left(\tfrac{\partial}{\partial \bar w_{q'}}-\tfrac{\partial K}{\partial \bar w_{q'}}\right)\left(\tfrac{\partial}{\partial \bar w_{j'}}-\tfrac{\partial K}{\partial \bar w_{j'}}\right)F_{\mathcal{S}}|_{z=w}\right) & F_{\mathcal{S}}(z,z) \end{array}\right],$$

$$C' = \left(\left(\tfrac{\partial}{\partial z_q}-\tfrac{\partial K}{\partial z_q}\right)\left(\tfrac{\partial}{\partial z_j}-\tfrac{\partial K}{\partial z_j}\right)\left(\tfrac{\partial}{\partial \bar w_{q'}}-\tfrac{\partial K}{\partial \bar w_{q'}}\right)\left(\tfrac{\partial}{\partial \bar w_{j'}}-\tfrac{\partial K}{\partial \bar w_{j'}}\right)F_{\mathcal{S}}|_{z=w}\right) \tag{76}$$

$$1\le j\le m\,, 1\le j\le q\le m\,, 1\le j'\le q'\le m\,.$$

In the formulas (74)–(76), we only take repeated holomorphic or anti-holomorphic derivatives of the potential K. Hence in an adapted frame of high order 2, the matrices simplify to the ones given in (6)–(9) when evaluated at $z=w=z_0$.

5.2. **Completion of the proof.** From (59), we obtain

$$D(0,H',x;z_0) = \frac{1}{\pi^{\binom{m+2}{2}}\det A\det\Lambda}\exp\left(-\langle\Lambda^{-1}(H'\oplus x), H'\oplus x\rangle\right)\ , \tag{77}$$

where Λ is given by (7). The formula in Theorem 4.4 then yields the expected density of critical points:

$$K(z_0) = \int_{\mathbb{C}^n} \| \det H \| D(0, H', x; z_0)\, dH'\, dx\,. \tag{78}$$

To complete the proof of Theorem 1, we need the formula for $\|\det H\|$. We now obtain the formula from (54) with $h = 1$ (normal coordinates at z_0) and

$$\begin{aligned} \xi_j &= H'_{j1} dz_1 + \cdots + H'_{jm} dz_m + H''_{j1} d\bar{z}_1 + \cdots + H''_{jm} d\bar{z}_m\,, \\ \bar{\xi}_j &= \overline{H''_{j1}} dz_1 + \cdots + \overline{H''_{jm}} dz_m + \overline{H'_{j1}} d\bar{z}_1 + \cdots + \overline{H'_{jm}} d\bar{z}_m\,. \end{aligned}$$

By (54), we see that $\|\det H\|$ is the determinant of the matrix H^c given by (38):

$$\| \det H \| \;=\; |\det H^c| \;=\; \left| \det \begin{bmatrix} H' & -x\Theta \\ -\bar{x}\bar{\Theta} & H'^* \end{bmatrix} \right|\,. \tag{79}$$

Theorem 1 now follows from (77)–(79). □

Remark: Our formulas for the A, B, C matrices differ slightly from those in [BSZ2] (and in our forthcoming paper [DSZ]). There, we lift the computations on a positive Hermitian line bundle to the associated circle bundle, which amounts to replacing $\rho^{\mathrm{diag}}_{e_L(z_0)}$ by ρ^{diag}_u, where $u = \|e_L(z_0)\|_h^{-1} e_L(z_0)$. The resulting formula for the density is the same in both approaches, since it is invariant when $\Delta(z_0)$ is multiplied by a scalar factor.

5.3. **Proof of Corollary 2.** We apply Theorem 1 with $\Theta = I$, which is the local formula for Θ in normal coordinates; equivalently $H''_{jq} = -\delta^q_j x$. Then

$$\begin{aligned} \| \det H \| &= \left| \det \begin{bmatrix} H' & H'' \\ \overline{H''} & \overline{H'} \end{bmatrix} \right| = \left| \det \begin{bmatrix} H' & -xI \\ -\bar{x}I & H'^* \end{bmatrix} \right| \\ &= \left| (\det H') \det(H'^* - |x|^2 H'^{-1}) \right| \\ &= \left| \det(H'H'^* - |x|^2 I) \right|\,. \end{aligned} \tag{80}$$

Therefore by (78)–(79),

$$K(z_0) = \frac{1}{\pi^{\binom{m+2}{2}} \det A \det \Lambda} \int \left| \det(H'H'^* - |x|^2 I) \right| e^{-\langle \Lambda^{-1}(H' \oplus x), H' \oplus x \rangle}\, dH'\, dx\,. \tag{81}$$

This completes the proof of Corollary 2. □

5.4. **Alternate viewpoint.** In this section, we give a different viewpoint to the proof of Theorem 1 that seems closer to the discussions in [Do].

Let us first consider the simpler case of critical points $\nabla f = 0$ with respect to the usual flat Euclidean gradient of real valued functions $f : \mathbb{R}^n \to \mathbb{R}$. The delta function on the critical set is then given by

$$C_f = \sum_{x: df(x) = 0} \delta_x,$$

where δ_x denotes the point mass at the point x, i.e. $\langle \delta_x, \varphi \rangle = \varphi(x)$ for test functions φ. The measure C_f is closely related to the pull back under $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ of the delta function

δ_0 at zero in $\mathbb{R}^n$. In general, let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth map all of whose zeros are non-degenerate in the sense that $\det DF_x \neq 0$ whenever $F(x) = 0$. Then,

$$F^*\delta_0 = \sum_{x:F(x)=0} \frac{\delta_x}{|\det DF_x|}. \tag{82}$$

If $F = \nabla f$ and f has only non-degenerate critical points, this becomes

$$(\nabla f)^*\delta_0 = \sum_{x:df(x)=0} \frac{\delta_x}{|\det D\nabla f(x)|}, \tag{83}$$

where $D\nabla f$ is the derivative of the map ∇f. The measures C_f and $\nabla f^*\delta_0$ are related by

$$C_f = |\det D\nabla f(x)|(\nabla f)^*\delta_0. \tag{84}$$

We now generalize to the local analogue of the case of concern in this paper, where $f : \mathbb{C}^m \to \mathbb{C}$ is holomorphic and ∇ is a smooth connection of type $(1,0)$, i.e. has the form $\nabla f = \partial f - f\partial K$ for f holomorphic. As before, we define

$$C_f = \sum_{z:\nabla f(z)=0} \delta_z, \quad z \in \mathbb{C}^m.$$

Relative to the global basis dz_j of $(1,0)$ forms on $\mathbb{C}^m$, we may express ∇f as the smooth map

$$\nabla f = (\nabla_{\frac{\partial}{\partial z_1}} f, \dots, \nabla_{\frac{\partial}{\partial z_m}} f) : \mathbb{C}^m \to \mathbb{C}^m, \tag{85}$$

where $\nabla_{\frac{\partial}{\partial z_j}} f = \frac{\partial}{\partial z_j} f - \frac{\partial K}{\partial z_j} f$.

Since ∇f is not holomorphic, its derivative $D^v\nabla f(z)$ is not a complex linear map on the complex tangent space $T_z\mathbb{C}^m \sim \mathbb{C}^m$, but rather is a linear map of the real tangent space, a real $2m$-dimensional vector space. At a critical point z_0, we express the derivative $D^v\nabla f(z) : T_z\mathbb{R}^{2m} \otimes \mathbb{C} \to T_0\mathbb{R}^{2m} \otimes \mathbb{C}$ in terms of the real basis $\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_k}$ of each complexified real tangent space:

$$D^v\nabla f = \begin{pmatrix} D_{\frac{\partial}{\partial z_k}} \nabla_{\frac{\partial}{\partial z_j}} f & D_{\frac{\partial}{\partial \bar{z}_k}} \nabla_{\frac{\partial}{\partial z_j}} f \\ & \\ D_{\frac{\partial}{\partial z_k}} \nabla_{\frac{\partial}{\partial \bar{z}_j}} \bar{f} & D_{\frac{\partial}{\partial \bar{z}_k}} \nabla_{\frac{\partial}{\partial \bar{z}_j}} \bar{f} \end{pmatrix}. \tag{86}$$

Since f is holomorphic, the off-diagonal blocks simplify to $\overline{\Theta f}$ and its complex conjugate. Thus, $D^v\nabla f$ is precisely the complex Hessian matrix H^c of (38), and hence it is the 'vertical derivative' from §3.1.

We then have

$$(\nabla f)^*\delta_0 = \sum_{z:df(z)=0} \frac{\delta_z}{|\det D^v\nabla f(z)|}, \tag{87}$$

and therefore

$$C_f = |\det D^v\nabla f(z)| \sum_{z:df(z)=0} (\nabla f)^*\delta_z,$$

where $D^v\nabla f$ is given in (86).

We now adapt these formulas to holomorphic sections $s \in H^0(M, L)$ and Chern connections for a Hermitian metric h, which reduce to the previous example in a local frame. In so doing, we justify the invariant interpretation of the Hermitian Hessian matrix H^c from §3.1.

We introduce local coordinates $z_1, \dots, z_n$ on M with Euclidean volume form dz as in Theorem 1. From an invariant point of view, the gradient map (30) is a section of $T^{*(1,0)} \otimes L$. Since the bundle is non-trivial, the delta-function at 0 in the previous calculation should be interpreted as the delta function δ_0 along the zero section of $T^{*(1,0)} \otimes L$, that is,

$$\langle \delta_0, \psi \rangle = \int_M \psi(z, 0) dV(z)$$

where $\psi \in \mathcal{C}^\infty(T^{*1,0} \otimes L)$. In a local frame e_L, the gradient map is given by (85) and the delta-function is just δ_0 on $\mathbb{C}^m$. This explains why the derivative D^v defined in the local discussion is the vertical part of the full derivative relative to the flat connection. In the setting of line bundles, (87) becomes

$$(\nabla s)^* \delta_0 = \sum_{z : \nabla s(z) = 0} \frac{\delta_z}{|\det D^v \nabla s(z)|}, \tag{88}$$

where

$$D^v \nabla s : T_z M \to T_z^{*1,0} \otimes L_z \tag{89}$$

is the vertical part of the derivative of (30). Taking the vertical part requires a connection on $T^{*1,0} \otimes L$, which we take to be the flat connection on our coordinate neighborhood (which we can do, since $\mathbf{k}^{\mathrm{crit}}_{\mathcal{S},\gamma,\nabla}$ is independent of the connection on M). As observed in §3.1, at a critical point, $D^v \nabla s(z)$ is independent of the choice of connection. The determinant is taken relative to the local Euclidean metric on M and the metric h on L_z.

It follows that

$$\mathbf{k}^{\mathrm{crit}}_{\mathcal{S},\gamma,\nabla}\, dz = \mathbf{E} C^{\nabla}_s = \mathbf{E}\big(|\det D^v \nabla s|\, \nabla s^* \delta_0\big). \tag{90}$$

In calculating $\mathbf{k}^{\mathrm{crit}}_{\mathcal{S},\gamma,\nabla}(z)$ we may fix z and introduce a local adapted frame e_L at z. Again writing $s = f e_L$, we have

$$\mathbf{k}^{\mathrm{crit}}_{\mathcal{S},\gamma,\nabla}(z) \;=\; \int_{\mathbb{C}^m} \int_{\mathbb{C}^d} |\det \sum a_j D^v \nabla f_j(z)| e^{i \mathrm{Re} \langle t, \sum_j a_j \nabla f_j(z) \rangle} e^{-|a|^2} da dt \tag{91}$$

In local coordinates, $D^v \nabla f(z)$ is the matrix H^c of (38).

We now calculate the da integral by making a change of variables. We consider the real linear map $\mathcal{J} := J^1_z$ of (48), which is locally written as

$$\mathcal{J}(a) = (\xi, H) := (\sum_j a_j \nabla f_j(z), \sum a_j D^v \nabla f_j(z)). \tag{92}$$

As mentioned above, for a positive line bundle the H matrix depends only on a complex $m \times m$ symmetric matrix (the holomorphic Hessian) and a complex scalar (which when multiplied by Θ, gives the mixed Hessian), so we may regard $\mathcal{J}$ as a map from $a \in \mathbb{C}^d$ into $(\xi, H) \in \mathbb{C}^m \times \mathrm{Sym}(m, \mathbb{C}) \times \mathbb{C}$ of dimension $\binom{m+2}{2}$. Since the integrand is a function only of ξ, H we may push forward the measure $e^{-|a|^2/2} da$ under J^1_z to obtain

$$\int_{\mathbb{C}^m} \int_{\mathbb{C}^m \times Sym(m,\mathbb{C}) \times \mathbb{C}} |\det H| e^{i \mathrm{Re} \langle t, \xi \rangle} J(\xi, H) d\xi dH dt, \tag{93}$$

where

$$\mathcal{J}_* e^{-|a|^2} da = J(\xi, H) d\xi dH, \;\text{ i.e. } J(\xi, H) = \int_{\mathcal{J}^{-1}(\xi, H)} e^{-|a|^2/2} d\dot{a}$$

where $d\dot{a}$ is the surface Lebesgue measure on the subspace $\mathcal{J}^{-1}(\xi, H)$. Evaluating the dt integral we obtain

$$(93) \;=\int_{\mathrm{Sym}(m,\mathbb{C})\times\mathbb{C}} |\det H| J(0,H)dH \tag{94}$$

To complete the proof, we need to evaluate $J(\xi, H)$. We claim that

$$J(0,H)=\frac{1}{\det A\;\det\Lambda}e^{-\langle\Lambda^{-1}H,H\rangle} \tag{95}$$

in our previous notation. This follows from general principles on pushing forward complex Gaussians under complex linear maps $F:\mathbb{C}^d\to\mathbb{C}^n$, whereby

$$F_* e^{-|a|^2}da=\gamma_{FF^*},$$

i.e.

$$J(\xi,H)=\frac{1}{\det\mathcal{J}\mathcal{J}^*}e^{\langle[\mathcal{J}\mathcal{J}^*]^{-1}(H,\xi),(H,\xi)\rangle}. \tag{96}$$

From

$$\langle\mathcal{J}(a),(\xi,H)\rangle=\{\langle\sum_j a_j\nabla f_j(z),\xi\rangle+\langle D^v\nabla f_j(z),H\rangle\}$$

we see that $\mathcal{J}^*:\mathbb{C}^m\times\mathrm{Sym}(m,\mathbb{C})\times\mathbb{C}\to\mathbb{C}^d$ is the map

$$\mathcal{J}^*(H,\xi)=(\{\langle\nabla f_j(z),\xi\rangle+\langle D^v\nabla f_j(z),H\rangle\}_{j=1}^d.$$

Hence $\mathcal{J}\mathcal{J}^*:\mathrm{Sym}(m,\mathbb{C})\times\mathbb{C}^m\to\mathrm{Sym}(m,\mathbb{C})\times\mathbb{C}^m$ is the map with block matrix form

$$\mathcal{J}\mathcal{J}^*(H,\xi)=\begin{pmatrix}A(\xi,H) & B(\xi,H)\\ B^*(\xi,H) & C(\xi,H)\end{pmatrix}$$

where

$$\begin{cases}A(\xi)=\sum_j\langle\nabla f_j(z),\xi\rangle\nabla f_j(z),\\ B(\xi,H)=\sum_j\langle D^v\nabla f_j(z),H\rangle\nabla f_j(z)\oplus\langle\nabla f_j(z),\xi\rangle D^v\nabla f_j(z),\\ C(H)=\sum_j\langle D^v\nabla f_j(z),H\rangle D^v\nabla f_j(z).\end{cases}$$

Summing in j we observe that

$$A=\sum_j\langle\nabla f_j(z),\xi\rangle\nabla f_j(z)=\nabla_z\nabla_{\bar w}\Pi(z,w)|_{z=w},\quad B=(\nabla_z\nabla^2_{\bar w}\Pi(z,w)|_{z=w},\;\nabla_z\Pi(z,w)|_{z=w}),$$

and

$$C=\begin{pmatrix}\nabla^2_z\nabla^2_{\bar w}\Pi(z,w)|_{z=w} & \nabla^2_z\Pi(z,w)|_{z=w}\\ \nabla^2_{\bar w}\Pi(z,w)|_{z=w} & \Pi(z,z)\end{pmatrix}. \tag{97}$$

Here, $|_{z=w}$ is shorthand for $\rho^{\mathrm{diag}}_{e_L(z)}$ (see (69).

To complete the proof of Theorem 1 we observe that when we set $\xi=0$, the quadratic form $\langle(\mathcal{J}\mathcal{J}^*)^{-1}(0,H),(0,H)\rangle$ equals $\langle\Lambda^{-1}(0,H),(0,H)\rangle$, where Λ is given by (7). $\square$

6. Exact formulas for Riemann surfaces

In this section, we derive exact formulas for the density of critical points on a Riemann surface with respect to any Hermitian line bundle, positive or not.

6.1. **Density of critical points on a Riemann surface: Proof of Theorem 3.** Let $(L,h) \to M$ be a Hermitian line bundle on a Riemann surface M with area form dV, and let $\mathcal{S}$ be a finite-dimensional subspace of $H^0(M,L)$ with the 2-jet spanning property, as in Theorem 3. Let $r = \frac{i}{2}\Theta_h/dV$, and let μ_1, μ_2 be the eigenvalues of ΛQ_r, where

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & -r^2 \end{pmatrix} .$$

We observe that μ_1, μ_2 have opposite signs since $\det \Lambda Q_r = -r^2 \det \Lambda < 0$. Let $\mu_2 < 0 < \mu_1$.

From the 1-dimensional case of Theorem 1, we have

$$\mathcal{K}^{\mathrm{crit}}_{\mathcal{S},h,V} = \frac{1}{\pi^3 A \det \Lambda} \int_{\mathbb{C}^2} \big|\, |H'|^2 - r^2 |x|^2 \,\big| e^{-\langle \Lambda^{-1}(H',x),(H',x)\rangle}\, dH'\, dx\,. \tag{98}$$

Writing $H = \begin{pmatrix} H' & x \end{pmatrix}$, we then have:

$$\begin{aligned} \mathcal{K}^{\mathrm{crit}}_{\mathcal{S},h,V} &= \frac{1}{\pi^3 A \det \Lambda} \int_{\mathbb{C}^2} \left| H Q_r H^* \right| \exp\left(-H\Lambda^{-1}H^*\right)\, dH \\ &= \frac{1}{\pi^3 A} \int_{\mathbb{C}^2} \left| H \Lambda^{\frac{1}{2}} Q_r \Lambda^{\frac{1}{2}} H^* \right| \exp\left(-HH^*\right)\, dH\,. \end{aligned}$$

We diagonalize $\Lambda^{\frac{1}{2}} Q_r \Lambda^{\frac{1}{2}}$, which has the same eigenvalues $\mu_1,\ \mu_2$ as ΛQ_r, to obtain

$$\begin{aligned} \mathcal{K}^{\mathrm{crit}}_{\mathcal{S},h,V} &= \frac{1}{\pi^3 A} \int_{\mathbb{C}^2} \left|\, \mu_1 |a|^2 + \mu_2 |b|^2 \,\right|\, e^{-|a|^2 - |b|^2}\, da\, db \\ &= \frac{1}{\pi A} \int_0^{+\infty} \int_0^{+\infty} |\mu_1 u + \mu_2 v|\, e^{-u-v} du\, dv \\ &= \frac{1}{\pi A \mu_1 |\mu_2|} \int_0^{+\infty} \int_0^{+\infty} |u - v|\, e^{-\mu_1^{-1} u - |\mu_2|^{-1} v} du\, dv \\ &= \frac{1}{\pi A \mu_1 |\mu_2|} \int_{-\infty}^{+\infty} \int_{\max\{w,0\}}^{+\infty} |w|\ \exp(|\mu_2|^{-1} w) \exp\left[-(\mu_1^{-1} + |\mu_2|^{-1}) u\right]\ du\, dw \\ &= \frac{1}{\pi A \mu_1 |\mu_2|}\, (\mathrm{I}) \ + \ \frac{1}{\pi A \mu_1 |\mu_2|}\, (\mathrm{II})\,, \end{aligned}$$

where

$$(\mathrm{I}) = \int_0^{+\infty} \int_w^{+\infty} w\ \exp(|\mu_2|^{-1} w) \exp\left[-(\mu_1^{-1} + |\mu_2|^{-1}) u\right]\ du\, dw = \frac{\mu_1^2}{\mu_1^{-1} + |\mu_2|^{-1}}\,,$$

and

$$(\mathrm{II}) = \frac{1}{2\pi} \int_{-\infty}^{0} \int_0^{+\infty} (-w)\ \exp(|\mu_2|^{-1} w) \exp\left[-(\mu_1^{-1} + |\mu_2|^{-1}) u\right]\ du\, dw = \frac{\mu_2^2}{\mu_1^{-1} + |\mu_2|^{-1}}\,.$$

This yields the desired formula. $\square$

6.2. **Index density: Proof of Corollary 4.** Critical points of a section s are (almost surely) of index ± 1. The above proof shows that the expected density of critical points of index 1 is given by

$$\mathcal{K}^{\rm crit}_{+} = \frac{1}{\pi A \mu_1 |\mu_2|}\,(\mathrm{I}) = \frac{1}{\pi A}\,\frac{\mu_1^2}{|\mu_1|+|\mu_2|}\ ,$$

while the expected density of critical points of index -1 is

$$\mathcal{K}^{\rm crit}_{-} = \frac{1}{\pi A \mu_1 |\mu_2|}\,(\mathrm{II}) = \frac{1}{\pi A}\,\frac{\mu_2^2}{|\mu_1|+|\mu_2|}\ .$$

Hence, the index density is given by

$$\mathcal{K}^{\rm crit}_{\rm index} := \mathcal{K}^{\rm crit}_{+} - \mathcal{K}^{\rm crit}_{-} = \frac{\mu_1+\mu_2}{\pi A} = \frac{Tr[\Lambda Q_r]}{\pi A}\ . \tag{99}$$

(Of course, (99) can also be obtained directly from (98) as an elementary second moment calculation.)

The critical points of s of index 1 are the saddle points of $|s|^2_h$, while those of index -1 are local maxima of $|s|^2_h$ in the case where L is positive, and are local minima of $|s|^2_h$ if L is negative. (The length $|s|$ cannot have positive local minima if L is positive, or maxima if L is negative.)

6.3. **Alternate proof of Theorem 3.** For simplicity, we assume that $r = 1$. From (98), we obtain

$$\begin{aligned}\mathcal{K}^{\rm crit}_{\mathcal{S},h,V} &= \frac{1}{\pi^3 A \det\Lambda}\int_{\mathbb{C}^2}\big|\,|H'|^2-|x|^2\,\big|\,e^{-\langle\Lambda^{-1}(H',x),(H',x)\rangle}\,dH'\,dx\\ &= \frac{1}{2\pi^4 A\ \det\Lambda}\int_{\mathbb{C}^2}\int_{\mathbb{R}}\int_{\mathbb{R}}\big|\lambda-|x|^2\big|\,e^{i\xi(\lambda-|H'|^2)}e^{-\langle\Lambda^{-1}(H',x),(H',x)\rangle}d\xi\,d\lambda\,dH'\,dx\,.\end{aligned}$$

Indeed, the ξ integral gives the δ-function at $\lambda = |H'|^2$, and the λ integral then gives (98).

We change variables to $\lambda' = \lambda - |x|^2$ to get (dropping the primes)

$$\frac{1}{2\pi^4 A\det\Lambda}\int_{\mathbb{C}^2}\int_{\mathbb{R}}\int_{\mathbb{R}}|\lambda|\,e^{i\xi(\lambda+|x|^2-|H|^2)}\exp\big(-\langle\Lambda^{-1}(H,x),(H,x)\rangle\big)\,d\xi\,d\lambda\,dH\,dx. \tag{100}$$

We now do the complex Gaussian $dHdx$ integral on $\mathbb{C}^2$. The quadratic form is

$$i\langle\xi,|x|^2-|H|^2\rangle - \langle\Lambda^{-1}(H,x),(H,x)\rangle = -\big\langle(\Lambda^{-1}+i\xi Q)(H,x),(H,x)\big\rangle\,.$$

The result is

$$\mathcal{K}^{\rm crit}_{\mathcal{S},h,V}(z) = \frac{1}{2\pi^2 A}\int_{\mathbb{R}}\int_{\mathbb{R}}|\lambda|\,e^{i\xi\lambda}\frac{1}{\det(I+i\xi\Lambda Q)}\,d\xi\,d\lambda\,. \tag{101}$$

Thus, in dimension one, $\mathcal{K}^{\rm crit}_{\mathcal{S},h,V}(z)$ depends only on the eigenvalues μ_1,μ_2 of ΛQ.

We first consider the $d\xi$ integral,

$$\mathcal{I}(\lambda) = \int_{\mathbb{R}}\frac{e^{i\xi\lambda}}{\det(I+i\xi\Lambda Q)}d\xi = \int_{\mathbb{R}}\frac{e^{i\xi\lambda}}{(1+i\xi\mu_1)(1+i\xi\mu_2)}d\xi = \frac{-1}{\mu_1\mu_2}\int_{\mathbb{R}}\frac{e^{i\xi\lambda}}{(\xi-i\mu_1^{-1})(\xi-i\mu_2^{-1})}d\xi.$$

We separately treat the cases $\lambda>0, \lambda<0$.

(i) $\lambda > 0$**:** In this case, we pick up the residue at the pole i/μ_1 in the upper half plane:

$$\mathcal{I}_+(\lambda) = \frac{-2\pi i}{\mu_1\mu_2}\mathrm{Res}_{i/\mu_1}\left[\frac{e^{i\xi\lambda}}{(\xi-i\mu_1^{-1})(\xi-i\mu_2^{-1})}\right] = \frac{-2\pi i}{\mu_1\mu_2}\frac{e^{-\lambda/\mu_1}}{(i\mu_1^{-1}-i\mu_2^{-1})} = \frac{2\pi\,e^{-\lambda/\mu_1}}{\mu_1-\mu_2} \tag{102}$$

(ii) $\lambda < 0$**:** In this case we pick up the residue at i/μ_2:

$$\mathcal{I}_-(\lambda) = \frac{2\pi i}{\mu_1\mu_2}\mathrm{Res}_{i/\mu_2}\left[\frac{e^{i\xi\lambda}}{(\xi - i\mu_1^{-1})(\xi - i\mu_2^{-1})}\right] = \frac{2\pi\, e^{\lambda/|\mu_2|}}{\mu_1 - \mu_2} \tag{103}$$

To complete the calculation, we need to evaluate

$$\begin{aligned}\int_{-\infty}^0 (-\lambda)\mathcal{I}_-(\lambda)d\lambda + \int_0^\infty \lambda\mathcal{I}_+(\lambda)d\lambda \\ &= \frac{2\pi}{\mu_1 - \mu_2}\left(\int_{-\infty}^0 (-\lambda)e^{\frac{\lambda}{|\mu_2|}}d\lambda + \int_0^\infty \lambda e^{-\frac{\lambda}{\mu_1}}d\lambda\right) \\ &= 2\pi\,\frac{\mu_2^2 + \mu_1^2}{|\mu_1| + |\mu_2|}\ .\end{aligned} \tag{104}$$

The desired formula follows from (101) and (104) □

We shall use this approach in [DSZ] for our computation of densities in higher dimensions.

6.4. **Exact formula for $\mathbb{CP}^1$: Proof of Corollary 5.** Since the critical point density with respect to the Fubini-Study metric is $SU(2)$ invariant and hence constant, it suffices to compute it at the point $(z_0 : 0) \in \mathbb{CP}^1$, using the local coordinate $z = z_1/z_0$ and the local frame e_N for $\mathcal{O}(N)$ corresponding to the homogeneous polynomial z_0^N on $\mathbb{C}^2$. We recall that the Szegö kernel is given by

$$\Pi_{H^0(\mathbb{CP}^1,\mathcal{O}(N))}(z,w) = \frac{N+1}{\pi}(1+z\bar{w})^N e_N(z)\otimes\overline{e_N(w)}\ .$$

(See, for example, [SZ, §1.3].) Since our formula is invariant when the Szegö kernel is multiplied by a constant, we can replace the above by the *normalized Szegö kernel*

$$\widetilde{\Pi}_N(z,w) := (1+z\bar{w})^N \tag{105}$$

in our computation.

Since

$$K(z) = -\log|e_N(z)|^2_{\mathrm{FS}} = N\log(1+|z|^2)\ ,$$

we have

$$K(0) = \frac{\partial K}{\partial z}(0) = \frac{\partial^2 K}{\partial z^2}(0) = 0\ ;$$

i.e., e_N is an adapted frame at $z = 0$. Hence when computing the (normalized) matrices $B_N,\ C_N$ for $H^0(\mathbb{CP}^1,\mathcal{O}(N))$, we can take the usual derivatives of $\widetilde{\Pi}_N$. Indeed, we have

$$\begin{aligned}\frac{\partial\widetilde{\Pi}_N}{\partial z} &= N(1+z\bar{w})^{N-1}\bar{w}\ , \\ \frac{\partial^2\widetilde{\Pi}_N}{\partial z\partial\bar{w}} &= N(1+z\bar{w})^{N-1} + N(N-1)(1+z\bar{w})^{N-2}z\bar{w}\ .\end{aligned}$$

It follows that

$$A_N(0) = (\ N\ )\ ,\quad B_N(0) = (\ 0\ \ 0\ )\ ,\quad \Lambda_N(0) = C_N(0) = \begin{pmatrix} 2N(N-1) & 0 \\ 0 & 1\end{pmatrix}\ . \tag{106}$$

We now apply Corollary 4. Since $r := \frac{i}{2}\Theta_h/dV = N$, where h is the Fubini-Study metric on $\mathcal{O}(N)$ and $dV = \omega_{\rm FS}$, the eigenvalues of $\Lambda_N(0)Q_r$ are given by:

$$\mu_1 = 2N(N-1), \quad \mu_2 = -N^2 \, .$$

Suppose that $N \geq 2$ so that $\mathcal{O}(N) \to \mathbb{CP}^1$ has the 2-jet spanning property. Theorem 3 then yields

$$\mathcal{K}^{\rm crit}_+ = \frac{1}{\pi}\,\frac{4(N-1)^2}{3N-2}\,, \quad \mathcal{K}^{\rm crit}_- = \frac{1}{\pi}\,\frac{N^2}{3N-2}\,. \tag{107}$$

Since $\mathcal{K}^{\rm crit}_\pm$ is constant by invariance of the metric and connection, and $\mathrm{Vol}(\mathbb{CP}^1) = \pi$, the desired formulas follow from (107).

If $N = 1$, then every section has exactly 1 critical point (of index 1), so the formula holds in this case too. □

Remark: For $N = 2$, it also turns out that almost all sections have exactly 2 critical points—one each of index $+1$ and -1. To see this, we first note that Theorem 5 says that the expected number of critical points in this case is 2. Since $\chi(\mathcal{O}(2) \otimes T^{*1,0}) = c_1(\mathcal{O}(2) \otimes T^{*1,0}) = 0$, the number of critical points of index 1 equals the number of critical points of index -1. Suppose that

$$s = (a + bz + cz^2)\, e^{\otimes 2}\, .$$

The critical point equation is:

$$(2a + b) + (2b + 2c)z + b|z|^2 = 0\, .$$

By Bézout's Theorem on $\mathbb{R}^2$, there are at most 4 critical points. Hence there are only two possibilities: (i) 2 critical points of index 1 and 2 of index -1; (ii) 1 critical point each of index 1 and of -1. Since the average number of critical points is 2, case (ii) almost always occurs.

However, for $N \geq 3$, one easily checks that the expected number of critical points, $\frac{5N^2-8N+4}{3N-2}$, is not an integer and hence the sections in $H^0(\mathbb{CP}^1, \mathcal{O}(N))$ cannot all have the same number of critical points.

6.4.1. *Metric dependence of the number of critical points.* The expected number of critical points $\mathcal{N}^{\rm crit}_{\mathcal{S},h} = \int_M \mathcal{K}^{\rm crit}_{\mathcal{S},h} dV$ of a section s of $H^0(M, L)$ (with the Hermitian Gaussian measure) depends on the metric h on L. This is true even for the case where $L = \mathcal{O}(1) \to \mathbb{CP}^1$ is the hyperplane section bundle over the projective line. To illustrate this dependence, we let $z = z_1/z_0$ denote the coordinate in the affine chart $\mathbb{C} \subset \mathbb{CP}^1$, and let $e_L = z_0 \in \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}) \approx H^0(\mathbb{CP}^1, L)$; then e_L is a local frame over $\mathbb{C}$. If we give L the standard Fubini-Study metric $h(e_L, e_L) = (1 + |z|^2)^{-1}$, then e_L has a critical point (maximum of $h(e_L, e_L)$) at 0 and no others. Furthermore, every section of $H^0(\mathbb{CP}^1, L)$ has exactly 1 critical point, so the expected number of critical points equals 1. Now let $p(z)$ be a polynomial of degree $k > 1$ with distinct roots, and consider the metric

$$\tilde h = \tilde h_0^{1-\varepsilon} h^{\varepsilon}\,, \qquad \tilde h_0 = (p^* h)^{\frac{1}{k}}\,,$$

where $\varepsilon > 0$. The metric $\tilde h$ has positive curvature (while the curvature of $\tilde h_0$ is semi-positive). Since the critical points of a section s coincide with the critical points of the function $\log |s|$, it suffices to consider

$$\log |e_L|_{\tilde h_0} = -\frac{1}{k} \log(1 + |p(z)|^2)\, .$$

This function has maxima at the k roots of p (and for generic p, has $k-1$ saddle points, by Morse theory) and hence has $2k-1$ critical points on $\mathbb{C}$. Therefore, the section $e_L = z_0$ has $2k-1$ critical points, and hence all nearby sections $z_0 + \delta z_1$ also have $2k-1$ critical points. As every section has at least one critical point, the expected number of critical points is greater than 1 for the metric $\tilde{h}$.

On the other hand, in [DSZ] we show that for any metric with positive curvature on $\mathbb{CP}^1$ (or more generally on any compact Kähler manifold), the expected number of critical points of $H^0(\mathbb{CP}^1, \mathcal{O}(N))$ has an asymptotic expansion in N^{-1}, where the first two terms are independent of the metric; see (15)–(16).

7. Morse index density: Proof of Theorem 6

We recall that the critical points of $\log|s|^2$ coincide with the critical points of ∇s and that they have Morse index $\geq m$. (Since almost all sections have only nondegenerate critical points, we make this assumption throughout.) Theorem 6 is an immediate consequence of Lemma 7.1 below and the proof of Theorem 1 and Corollary 2.

Recall that the Hermitian metric on $T^{1,0}_M$ is given by the curvature

$$\Theta = -\nabla'\nabla'' \log h \in T^{1,0^*}_M \otimes T^{0,1^*}_M = T^{1,0^*}_M \otimes \overline{T^{1,0^*}_M} \ .$$

We let $\Theta^* \in T^{1,0}_M \otimes \overline{T^{1,0}_M}$ denote the dual metric on $T^{1,0^*}_M$.

Lemma **7.1.** *Let $(L, h) \to M$ be a positive holomorphic line bundle, and let $z_0 \in M$ be a nondegenerate critical point of $s \in H^0(M, L)$. Then the Morse index of $\log|s|^2_h$ at z_0 equals $m + \mathrm{index}_{z_0}(S\Theta^*\overline{S} - \Theta)$, where*

$$S = \nabla'\nabla' \log|s|^2_h \in T^{1,0^*}_M \otimes T^{1,0^*}_M .$$

Hence at a critical point, the topological index of s is $(-1)^{m+n}$, where n is the Morse index of $\log|s|_h$.

Proof. Let z_0 be a nondegenerate critical point of $\log|s|^2$ at z_0. Let $z_j = x_j + iy_j$, $1 \leq j \leq m$, be normal coordinates at z_0. Note that $\Theta_{z_0} = \sum_{j=1}^m dz_j \otimes d\bar{z}_j$. Thus

$$\mathrm{index}_{z_0}(S\Theta^*\overline{S} - \Theta) = \mathrm{index}(SS^* - I) \ ,$$

where S now denotes the symmetric matrix

$$\big(S_{jk}\big) = \left(\frac{\partial^2}{\partial z_j \partial z_k} \log|s|^2_h\right)_{z_0} \ .$$

Conjugating the Hessian matrix

$$\begin{pmatrix} \left(\frac{\partial^2}{\partial x_j \partial x_k} \log|s|^2_h\right) & \left(\frac{\partial^2}{\partial x_j \partial y_k} \log|s|^2_h\right) \\ \left(\frac{\partial^2}{\partial y_j \partial x_k} \log|s|^2_h\right) & \left(\frac{\partial^2}{\partial y_j \partial y_k} \log|s|^2_h\right) \end{pmatrix}$$

with the unitary matrix

$$\frac{1}{\sqrt{2}} \begin{pmatrix} I & iI \\ iI & I \end{pmatrix} \ ,$$

we get

$$2\begin{pmatrix} \left(\frac{\partial^2}{\partial z_j\partial \bar z_k}\log|s|_h^2\right) & \left(i\frac{\partial^2}{\partial z_j\partial z_k}\log|s|_h^2\right) \\ \left(-i\frac{\partial^2}{\partial \bar z_j\partial \bar z_k}\log|s|_h^2\right) & \left(\frac{\partial^2}{\partial z_j\partial \bar z_k}\log|s|_h^2\right)\end{pmatrix}\ . \tag{108}$$

Write $s = f\,e_L$, where e_L is a local frame for L. Since

$$\log|s|_h^2 = \log|f|^2 + \log h$$

and $\{z_j\}$ are normal coordinates, we have

$$\frac{\partial^2}{\partial z_j\partial \bar z_k}\log|s|_h^2 = -\delta_j^k\ .$$

Thus, (108) becomes:

$$2\begin{pmatrix} -I & iS \\ -iS^* & -I\end{pmatrix}\ . \tag{109}$$

Let λ be an eigenvalue of

$$\widehat{S} := \begin{pmatrix} -I & iS \\ -iS^* & -I\end{pmatrix}\ ,$$

Then

$$0 = \det\begin{pmatrix} (-1-\lambda)I & iS \\ -iS^* & (-1-\lambda)I\end{pmatrix} = \det\big[(1+\lambda)^2 I - SS^*\big].$$

Therefore, $\mu := (1+\lambda)^2$ is an eigenvalue of SS^*. On the other hand, each eigenvalue μ of SS^* yields the pair of eigenvalues $-1\pm\sqrt{\mu}$ of $\widehat{S}$. Hence the number of negative eigenvalues of the Hessian of $\log|s|_h^2$ equals m plus the number of eigenvalues of SS^* that are less than 1. $\square$

Remark: If the line bundle $(L,h)\to M$ instead has negative curvature, then the Morse index of $\log|s|_h^2$ at a critical point $z_0\in M$ equals $\mathrm{index}_{z_0}(S\Theta^*\overline{S}-\Theta)$. To see this, we choose normal coordinates at z_0 with respect to the metric $-\Theta$. Then,

$$\mathrm{index}_{z_0}(S\Theta^*\overline{S}-\Theta) = \mathrm{index}(I-SS^*)\ .$$

This time, the Morse index of $\log|s|_h^2$ is the number of negative eigenvalues of

$$\widehat{S} := \begin{pmatrix} I & iS \\ -iS^* & I\end{pmatrix}.$$

Thus, each eigenvalue μ of SS^* corresponds to the pair of eigenvalues $1\pm\sqrt{\mu}$ of $\widehat{S}$, and the conclusion follows as above.

References

[AGV] V. I. Arnold, S. M. Gusein-Zade and A. N. Varchenko, *Singularities of differentiable maps. Vol. II. Monodromy and asymptotics of integrals*, Monographs in Math. 83, Birkhäuser, Boston, 1988.

[AD] S. Ashok and M. R. Douglas, Counting Flux Vacua, arXiv preprint hep-th/0307049(2003).

[BW] J. Bagger and E. Witten, Quantization of Newton's constant in certain supergravity theories. *Phys. Lett. B* 115 (1982), no. 3, 202–206.

[Ba] T. Banks, Heretics of the false vacuum: Gravitational effects on and of vacuum decay. arXiv preprint hep-th/0211160 (2002).

[BSZ1] P. Bleher, B. Shiffman and S. Zelditch, Universality and scaling of correlations between zeros on complex manifolds, *Invent. Math.* 142 (2000) 2, 351-395.

[BSZ2] P. Bleher, B. Shiffman and S. Zelditch, Universality and scaling of zeros on symplectic manifolds, in *Random Matrix Models and Their Applications*, ed. P. Bleher and A.R. Its, MSRI publications 40, Cambridge Univ. Press (2001), 31-70.

[Bo] R. Bott, On a theorem of Lefschetz. *Michigan Math. J.* 6 (1959), 211–216.

[BF] P. Breitenlohner and D. Z. Freedman, Stability in gauged extended supergravity, *Ann. Phys.* 144 (1982) 249-281.

[CHSW] P. Candelas, G. Horowitz, A. Strominger, and E. Witten, Vacuum configurations for superstrings. *Nuclear Phys. B* 258 (1985), no. 1, 46–74.

[CDL] S. R. Coleman and F. De Luccia, Gravitational Effects On And Of Vacuum Decay, *Phys. Rev. D* **21** (1980), 3305.

[Di] R. Dijkgraaf, Fields, strings and duality. In: *Symètries quantiques* (Les Houches, 1995), 3–147, North-Holland, Amsterdam, 1998.

[Do] M. R. Douglas, The statistics of string/M theory vacua. *J. High Energy Phys.* 2003, no. 5, 046 (arXiv preprint hep-th/0303194).

[DSZ] M. R. Douglas, B. Shiffman and S. Zelditch, Critical points and supersymmetric vacua, II: Asymptotics and extremal metrics.

[Fe] H. Federer, *Geometric Measure Theory*, Springer, New York, 1969.

[Fr1] D. S. Freed, *Five lectures on supersymmetry.* American Mathematical Society, Providence, RI, 1999.

[Fr2] D. S. Freed, Special Kähler manifolds. *Comm. Math. Phys.* 203 (1999), no. 1, 31–52 (arXiv preprint hep-th/9712042).

[GKP] S. B. Giddings, S. Kachru and J. Polchinski, Hierarchies from fluxes in string compactifications, *Phys. Rev,* D **66**, 106006 (2002) (arXiv preprint hep-th/0105097).

[GSW1] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring theory. Vol. 1. Introduction.* Second edition. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, 1988.

[GSW2] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring theory. Vol. 2. Loop amplitudes, anomalies and phenomenology.* Second edition. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, 1988.

[GH] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley-Interscience, N.Y. (1978).

[Ho] L. Hörmander, *The analysis of linear partial differential operators. I. Distribution theory and Fourier analysis.* Classics in Mathematics. Springer-Verlag, Berlin, 2003.

[IAS] *Quantum fields and strings: a course for mathematicians. Vol. 1, 2.* Material from the Special Year on Quantum Field Theory held at the Institute for Advanced Study, Princeton, NJ, 1996–1997. Edited by Pierre Deligne, Pavel Etingof, Daniel S. Freed, Lisa C. Jeffrey, David Kazhdan, John W. Morgan, David R. Morrison and Edward Witten. American Mathematical Society, Providence, RI; Institute for Advanced Study (IAS), Princeton, NJ, 1999. Vol. 1: xxii+723 pp.; Vol. 2: pp. i–xxiv and 727–1501

[KKLT] S. Kachru, R. Kallosh, A. Linde and S. P. Trivedi, De Sitter vacua in string theory, *Phys. Rev. D* **68**, 046005 (2003); arXiv preprint hep-th/0301240.

[Mi] J. Milnor, *Singular points of complex hypersurfaces.* Annals of Mathematics Studies, No. 61 Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo 1968.

[SZ] B. Shiffman and S. Zelditch, Distribution of zeros of random and quantum chaotic sections of positive line bundles, *Commun. Math. Phys.* 200 (1999), 661–683.

[St] A. Strominger, Special Geometry, *Comm. Math. Phys.* **133**, 163 (1990).

[We] S. Weinberg, *The quantum theory of fields. Vol. III. Supersymmetry*, Cambridge University Press, Cambridge, 2000.

[WB] J. Wess and J. Bagger, *Supersymmetry and supergravity.* Second edition. Princeton Series in Physics. Princeton University Press, Princeton, NJ, 1992.

Rutgers, IHES and Caltech
E-mail address: mrd@physics.rutgers.edu

Department of Mathematics, Johns Hopkins University, Baltimore, MD 21218, USA
E-mail address: shiffman@math.jhu.edu

Department of Mathematics, Johns Hopkins University, Baltimore, MD 21218, USA
E-mail address: zelditch@math.jhu.edu